Physics 523, Quantum Field Theory II

Homework 1

Due Wednesday, 14th January 2004

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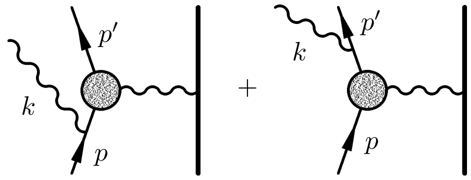
Bremsstrahlung

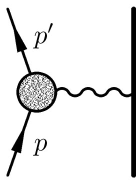
a) We showed that in the low energy limit, the amplitude for Bremsstrahlung,

$i\widetilde{\mathcal{M}} =$



$$= e\overline{u}(p')\mathcal{M}_o(p',p)u(p)\left(\frac{p'^{\mu}}{p'\cdot k} - \frac{p^{\mu}}{p\cdot k}\right)\epsilon^*_{\mu}, \tag{1.1}$$

can be written in terms of the amplitude for the process without bremsstrahlung which given in terms of the relativistically corrected amplitude $\mathcal{M}_o(p',p)$,



$$= i\overline{u}(p')\mathcal{M}_o(p',p)u(p).$$

We are to verify that (1.1) does indeed vanish when $\epsilon_\mu = k_\mu$. This can be easily seen by direct calculation.

$$\begin{aligned}
i\widetilde{\mathcal{M}} &= e\overline{u}(p')\mathcal{M}_o(p',p)u(p)\left(\frac{p'^{\mu}}{p'\cdot k} - \frac{p^{\mu}}{p\cdot k}\right)k_\mu,\\
&= e\overline{u}(p')\mathcal{M}_o(p',p)u(p)\left(\frac{p'^{\mu}k_\mu}{p'\cdot k} - \frac{p^{\mu}k_\mu}{p\cdot k}\right),\\
&= e\overline{u}(p')\mathcal{M}_o(p',p)u(p)\left(1-1\right),
\end{aligned}$$

$$\boxed{\therefore\, i\widetilde{\mathcal{M}}^{\mu}k_\mu = 0.} \tag{1.2}$$

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b) While in the soft photon limit this amplitude is consistent with current conservation, we will show that it fails in complete generality. To see this, let us consider the full amplitude for the two diagrams,

$$i\mathcal{M} = e\overline{u}(p')\left\{\mathcal{M}_o(p',p-k)\frac{\not{p}-\not{k}+m}{(p-k)^2-m^2}\gamma^\mu\epsilon^*_\mu(k) + \gamma^\mu\epsilon^*_\mu\frac{\not{p}'+\not{k}+m}{(p'+k)^2-m^2}\mathcal{M}_o(p'+k,p)\right\}u(p). \tag{1.3}$$

Now, recalling our work with the Dirac equation (and its conjugate) we see that,

$$(\not{p}+m)\gamma^\mu u(p) = 2p^\mu u(p), \quad \text{and} \quad \overline{u}(p')\gamma^\mu(\not{p}'+m) = \overline{u}(p')2p'^\mu.$$

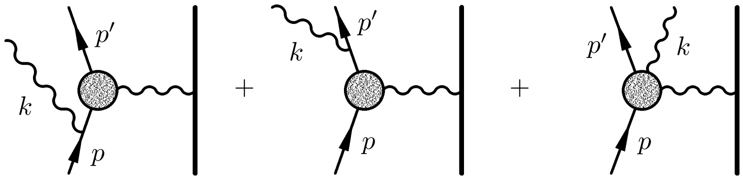
Combining this result with simple kinematics for the case where $\epsilon_\mu = k_\mu$ we have

$$\begin{aligned}
i\mathcal{M} &= e\overline{u}(p')\left\{k_\mu\frac{2p'^\mu+\gamma^\mu\not{k}}{2p'\cdot k}\mathcal{M}_o(p'+k,p) - \mathcal{M}_o(p',p-k)\frac{2p^\mu-\not{k}\gamma^\mu}{2p\cdot k}k_\mu\right\}u(p),\\
&= e\overline{u}(p')\left\{\frac{2p'\cdot k+\not{k}\not{k}}{2p'\cdot k}\mathcal{M}_o(p'+k,p) - \mathcal{M}_o(p',p-k)\frac{2p\cdot k-\not{k}\not{k}}{2p\cdot k}\right\}u(p),\\
&= e\overline{u}(p')\left[\mathcal{M}_o(p'+k,p) - \mathcal{M}_o(p',p-k)\right]u(p),
\end{aligned}$$

Now, this result cannot be vanishing for an arbitrary photon energy k. It is certainly the case that $\mathcal{M}_o(p'+k,p) = \mathcal{M}_o(p',p-k)$ to the order $\mathcal{O}(1/k)$ but certainly not in general. We will have to add an additional diagram to see true current conservation.

c) We can improve our estimate of the amplitude to emit a photon by Bremsstrahlung by adding a third diagram in which the photon is emitted from the 'gut' of the reaction with some amplitude $i\mathcal{M}_3 = e\overline{u}(p')\epsilon^*_\mu S^\mu u(p)$. Adding this diagram, we arrive have

$i\mathcal{M}_{\text{total}} =$



$$= \epsilon^*_\mu(k)e\overline{u}(p')\left\{\frac{2p'^\mu + \gamma^\mu \not{k}}{2p'\cdot k}\mathcal{M}_o(p'+k,p) - \mathcal{M}_o(p',p-k)\frac{2p^\mu - \not{k}\gamma^\mu}{2p\cdot k} - S^\mu\right\}u(p). \tag{1.4}$$

Therefore, we see that gauge invariance which demands that $k_\mu \mathcal{M}^\mu_{\text{total}} = 0$ implies that

$$k_\mu \mathcal{M}^\mu_{\text{total}} = 0 = e\overline{u}(p')k_\mu\left\{\frac{2p'^\mu + \gamma^\mu \not{k}}{2p'\cdot k}\mathcal{M}_o(p'+k,p) - \mathcal{M}_o(p',p-k)\frac{2p^\mu - \not{k}\gamma^\mu}{2p\cdot k} - S^\mu\right\}u(p),$$
$$= e\overline{u}(p')[\mathcal{M}_o(p'+k,p) - \mathcal{M}_o(p',p-k) - k_\mu S^\mu]u(p),$$

Therefore we see at once that gauge invariance implies that

$$k_\mu S^\mu = \mathcal{M}_o(p'+k,p) - \mathcal{M}_o(p',p-k). \tag{1.5}$$

d) Let us expand in derivatives of $\mathcal{M}_o$'s on the right. Doing this, we see that

$$k_\mu S^\mu = \frac{\partial}{\partial p'^\mu}\mathcal{M}_o(p',p)k^\mu + \frac{\partial}{\partial p^\mu}\mathcal{M}_o(p',p)k^\mu. \tag{1.6}$$

This implies that

$$S^\mu = \left(\frac{\partial}{\partial p'_\mu} + \frac{\partial}{\partial p_\mu}\right)\mathcal{M}_o(p',p) + \text{ divergenceless term.}$$

Now, At low energy, all divergenceless terms will go to zero and so our approximation of

$$S^\mu = \left(\frac{\partial}{\partial p'_\mu} + \frac{\partial}{\partial p_\mu}\right)\mathcal{M}_o(p',p), \tag{1.7}$$

is good to $\mathcal{O}(1)$.

Returning to the process of soft Bremsstrahlung, we see that the total amplitude to order $\mathcal{O}(1)$ can be written as

$$i\mathcal{M}_{\text{total}} = e\overline{u}(p')\left\{\frac{p'^\mu}{p'\cdot k} - \frac{p^\mu}{p\cdot k} - \frac{\partial}{\partial p'_\mu} - \frac{\partial}{\partial p_\mu}\right\}\epsilon^*_\mu(k)\mathcal{M}_o(p',p)u(p). \tag{1.8}$$

Physics 523, Quantum Field Theory II
Homework 2
Due Wednesday, 21st January 2004

Jacob Lewis Bourjaily

1. Feynman Parametrization

We are to prove *Feynman's Formula*,

$$\frac{1}{A_1\cdots A_n} = \int_0^1 dx_1\cdots dx_n \delta^{(n)}\left(\sum_{i=1}^n x_i - 1\right)\frac{(n-1)!}{[x_1A_1+\cdots+x_nA_n]^n}. \tag{1.1}$$

We will prove this result by induction. First, we will show that

$$\frac{1}{A_1A_2} = \int_0^1 dx_1dx_2\delta^{(2)}(x_1+x_2-1)\frac{1}{[x_1A_1+x_2A_2]^2}. \tag{1.2}$$

This integral can be simplified by using the dirac δ-function so that,

$$\int_0^1 dx_1dx_2\delta^{(2)}(x_1+x_2-1)\frac{1}{[x_1A_1+x_2A_2]^2} = \int_0^1 dx_1\frac{1}{[x_1A_1+(1-x_1)A_2]^2}. \tag{1.3}$$

We will solve this integral by making the substitution $u\equiv(x_1A_1+(1-x_1)A_2)$ so that $du=(A_1-A_2)dx_1$. Substituting u in the integral above and noting the change in the limits of integration we see immediately that

$$\frac{1}{A_1A_2} = \int_{A_2}^{A_1}\frac{du}{(A_1-A_2)}\frac{1}{u^2} = \frac{1}{(A_1-A_2)}\left(-\frac{1}{u}\right)\Bigg|_{A_2}^{A_1} = \frac{1}{A_1-A_2}\left(\frac{1}{A_2}-\frac{1}{A_1}\right), \tag{1.4}$$

$$\therefore \int_0^1 dx_1dx_2\delta^{(2)}(x_1+x_2-1)\frac{1}{[x_1A_1+x_2A_2]^2} = \frac{1}{A_1A_2}. \tag{1.5}$$

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Before we complete our proof, let us prove the lemma,

$$\frac{1}{A_1A_2^n} = \int_0^1 dx_1dx_2\delta^{(2)}(x_1+x_2-1)\frac{nx_2^{n-1}}{[x_aA_1+x_2A_2]^{n+1}}. \tag{1.6}$$

This lemma will be proved by induction. We have shown that for $n=1$ equation (1.6) holds. Now, let us suppose that (1.6) is true for some exponent $m\geq 1$. We must show that this implies that (1.6) is satisfied for $m+1$. So our induction hypothesis is given by

$$\frac{1}{A_1A_2^m} = \int_0^1 dx_1dx_2\delta^{(2)}(x_1+x_2-1)\frac{mx_2^{m-1}}{[x_aA_1+x_2A_2]^{m+1}}. \tag{1.7}$$

Let us differentiate both side of equation (1.7) with respect to A_2. This becomes

$$-m\frac{1}{A_1A_2^{m+1}} = -\int dx_1dx_2\delta^{(2)}(x_1+x_2-1)\frac{m(m+1)x_2^{m-1}x_2}{[x_1A_1+x_2A_2]^{m+2}}, \tag{1.8}$$

$$\therefore \frac{1}{A_1A_2^{m+1}} = \int dx_1dx_2\delta^{(2)}(x_1+x_2-1)\frac{(m+1)x_2^m}{[x_1A_1+x_2A_2]^{m+2}}. \tag{1.9}$$

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Now we are ready to complete the entire proof. Because we have shown that Feynman's formula is true for $\frac{1}{A_1A_2}$, we may prove by induction to $\frac{1}{A_1\cdots A_n}$. Let us assume therefore that Feynman's formula is valid for some $n=m\geq 2$. We must show that it is valid for $m+1$.

We will begin this proof by direct calculation. For this derivation, we will use the following notational conveniences:

$$\mathscr{U} \equiv (x_1 A_1 + \cdots + x_n A_m), \quad u_i \equiv (1 - u_{m+1}) x_i, \quad du_i \equiv (1 - x_{m+1}) dx_i \text{ for } i \in [1, m].$$

Note that u_{m+1} is an ordinary integration variable and is not set by the above. By our induction hypothesis, we have that

$$\frac{1}{A_1 \cdots A_m} = \int_0^1 dx_1 \cdots dx_m \delta^{(m)} \left(\sum_{i=1}^{m} x_i - 1 \right) \frac{(m-1)!}{[x_1 A_1 + \cdots + x_m A_m]^m}.$$

We also note the property of the Dirac δ-functional that $\delta(f(x)/a) = a\delta(f(x))$. Now, let us make the following calculation

$$\begin{aligned}
\frac{1}{A_1 \cdots A_{m+1}} &= \frac{1}{A_{m+1}} \frac{1}{A_1 \cdots A_m}, \\
&= \frac{1}{A_{m+1}} \int_0^1 dx_1 \cdots dx_m \delta \left(\sum_{i=1}^{m} x_i - 1 \right) \frac{(m-1)!}{[x_1 A_1 + \cdots + x_m A_m]^m}, \\
&= \int_0^1 dx_1 \cdots dx_m \delta \left(\sum_{i=1}^{m} x_i - 1 \right) (m-1)! \frac{1}{\mathscr{U}^m} \frac{1}{A_{m+1}}, \\
&= \int_0^1 dx_1 \cdots dx_m \delta \left(\sum_{i=1}^{m} x_i - 1 \right) (m-1)! \int_0^1 du_{m+1} \frac{m(1-u_{m+1})^{m-1}}{[(1-u_{m-1})\mathscr{U} + u_{m+1} A_{m+1}]^{m+1}}, \\
&= \int_0^{1-u_{m+1}} du_1 \cdots \; du_m \delta \left(\sum_{i=1}^{m} x_i - 1 \right) \frac{m!}{(1-x_{m+1})^m} \int_0^1 du_{m+1} \frac{(1-u_{m+1})^{m-1}}{[u_1 A_1 + \cdots + u_m A_m + u_{m+1} A_{m+1}]^{m+1}}, \\
&= \int_0^1 du_{m+1} \int_0^{1-u_{m+1}} du_1 \cdots du_m \delta \left(\sum_{i=1}^{m} \frac{u_i}{(1-u_{m+1})} - 1 \right) \frac{m!}{(1-u_{m+1})[u_1 A_1 + \cdots + u_{m+1} A_{m+1}]^{m+1}}, \\
&= \int_0^1 du_{m+1} \int_0^{1-u_{m+1}} du_1 \cdots du_m \delta \left(\sum_{i=1}^{m+1} u_i - 1 \right) \frac{m!}{[u_1 A_1 + \cdots + u_{m+1} A_{m+1}]^{m+1}}.
\end{aligned}$$

We note that because of the δ-functional within the integral (and because u_{m+1} is always positive), when the domain of the interior integral is extended to 1 the integral will not pick up any additional contribution. So we may put the integral above into a more symmetric form,

$$\frac{1}{A_1 \cdots A_{m+1}} = \int_0^1 du_1 \cdots du_{m+1} \delta \left(\sum_{i=1}^{m+1} u_i - 1 \right) \frac{m!}{[u_1 A_1 + \cdots + u_{m+1} A_{m+1}]^{m+1}}. \tag{1.10}$$

Therefore, by induction on m we see that for all values $n \geq 2$,

$$\frac{1}{A_1 \cdots A_n} = \int_0^1 dx_1 \cdots dx_n \delta^{(n)} \left(\sum_{i=1}^{n} x_i - 1 \right) \frac{(n-1)!}{[x_1 A_1 + \cdots + x_n A_n]^n}. \tag{1.11}$$

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2. Loop Integrals

a) We are to demonstrate that

$$\int \frac{d^4 \ell}{(2\pi)^4} \frac{1}{[\ell^2 - \Delta]^m} = \frac{i(-1)^m}{(4\pi)^2} \frac{1}{(m-1)(m-2)} \frac{1}{\Delta^{m-2}} \quad \text{for } m > 2.$$

To compute this integral, we will first note that the two poles, at $\ell = \pm\sqrt{\Delta}$, are covered by the same contour in the complex ℓ^0 plane when the contour is analytically extended to the imaginary axis. Therefore, without loss of generality, we may make the substitution $\ell = i\ell_E$.

Doing this, we may compute directly. Note the substitution $u \equiv \ell_E^2 + \Delta$ in the fifth line. Also notice that the derivation is only valid for $m > 2$ because the integral will diverge for $m \leq 2$.

$$\begin{aligned}
\int \frac{d^4\ell}{(2\pi)^4} \frac{1}{[\ell^2 - \Delta]^m} &= \frac{i}{(2\pi)^4} \int d^4\ell_E \frac{1}{[-\ell_E^2 - \Delta]^m}, \\
&= \frac{i(-1)^m}{(2\pi)^4} \int d^4\ell_E \frac{1}{[\ell_E^2 + \Delta]^m}, \\
&= \frac{i(-1)^m}{(2\pi)^4} \int d\Omega_4 \int_0^\infty d\ell_E \frac{\ell_E^3}{[\ell_E^2 + \Delta]^m}, \\
&= \frac{2i(-1)^m}{(4\pi)^2} \int_0^\infty d\ell_E \frac{\ell_E^3}{[\ell_E^2 + \Delta]^m}, \\
&= \frac{2i(-1)^m}{(4\pi)^2} \int_\Delta^\infty \frac{du}{2\ell_E} \frac{\ell_E^3}{u^m}, \\
&= \frac{i(-1)^m}{(4\pi)^2} \int_\Delta^\infty du \frac{u - \Delta}{u^m}, \\
&= \frac{i(-1)^m}{(4\pi)^2} \left(\frac{1}{(m-1)} \frac{\Delta}{u^{m-1}} - \frac{1}{(m-2)} \frac{1}{u^{m-2}} \right) \Bigg|_\Delta^\infty, \\
&= \frac{i(-1)^m}{(4\pi)^2} \left(\frac{1}{(m-2)} \frac{1}{\Delta^{m-2}} - \frac{1}{(m-1)} \frac{1}{\Delta^{m-2}} \right),
\end{aligned}$$

$$\therefore \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{[\ell^2 - \Delta]^m} = \frac{i(-1)^m}{(4\pi)^2} \frac{1}{(m-1)(m-2)} \frac{1}{\Delta^{m-2}} \quad \text{for } m > 2. \tag{2.1}$$

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b) We are to demonstrate that

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^2}{[\ell^2 - \Delta]^m} = \frac{i(-1)^{m-1}}{(4\pi)^2} \frac{2}{(m-1)(m-2)(m-3)} \frac{1}{\Delta^{m-3}} \quad \text{for } m > 3.$$

To prove this equality we will proceed similarly to part (a) above. Like before, we note that the two residues, at $\ell = \pm\sqrt{\Delta}$, are covered by the same branch cut in the complex plane when the contour integral is analytically continued to the imaginary axis. Therefore, we will make the substitution $\ell = i\ell_E$. When computing the integral explicitly below, note the substitution $u \equiv \ell_E^2 + \Delta$. Also, notice that for $m \leq 3$ the integral will diverge. We will proceed directly.

$$\begin{aligned}
\int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^2}{[\ell^2 - \Delta]^m} &= \frac{i}{(2\pi)^4} \int d^4\ell_E \frac{-\ell_E^2}{[-\ell_E^2 - \Delta]^m}, \\
&= \frac{i(-1)^{m-1}}{(2\pi)^4} \int d\Omega_4 \int_0^\infty d\ell_E \frac{\ell_E^5}{[\ell_E^2 + \Delta]^m}, \\
&= \frac{2i(-1)^{m-1}}{(4\pi)^2} \int_\Delta^\infty \frac{du}{2\ell_E} \frac{\ell_E^5}{u^m}, \\
&= \frac{i(-1)^{m-1}}{(4\pi)^2} \int_\Delta^\infty du \frac{\ell_E^4}{u^m}, \\
&= \frac{i(-1)^{m-1}}{(4\pi)^2} \int_\Delta^\infty du \frac{(u^2 - 2\Delta u + \Delta^2)}{u^m}, \\
&= \frac{i(-1)^{m-1}}{(4\pi)^2} \left(-\frac{1}{(m-3)} \frac{1}{u^{m-3}} + \frac{1}{(m-2)} \frac{2\Delta}{u^{m-2}} - \frac{1}{(m-1)} \frac{\Delta^2}{u^{m-1}} \right) \Bigg|_\Delta^\infty, \\
&= \frac{i(-1)^{m-1}}{(4\pi)^2} \frac{1}{\Delta^{m-3}} \left(\frac{1}{(m-3)} - \frac{2}{(m-2)} + \frac{1}{(m-1)} \right),
\end{aligned}$$

$$\therefore \int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^2}{[\ell^2 - \Delta]^m} = \frac{i(-1)^{m-1}}{(4\pi)^2} \frac{2}{(m-1)(m-2)(m-3)} \frac{1}{\Delta^{m-3}} \quad \text{for } m > 3. \tag{2.2}$$

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c) Let us prove the identity

$$\int \frac{d^4\ell}{(2\pi)^4}\left(\frac{\ell^2}{[\ell^2-\Delta]^3}-\frac{\ell^2}{[\ell^2-\Delta_\Lambda]^3}\right)=\frac{i}{(4\pi)^2}\ln\left(\frac{\Delta_\Lambda}{\Delta}\right).$$

To prove this identity we will differentiate both sides with respect to Δ. Doing this, the right hand side trivially becomes (noting the definition of Δ_Λ),

$$\frac{\partial}{\partial\Delta}\left\{\frac{i}{(4\pi)^2}\ln\left(\frac{\Delta_\Lambda}{\Delta}\right)\right\}=\frac{i}{(4\pi)^2}\frac{\Delta}{\Delta_\Lambda}\left(-\frac{z\Lambda^2}{\Delta^2}\right)=\frac{i}{(4\pi)^2}\frac{-z\Lambda^2}{\Delta_\Lambda\Delta}. \tag{2.3}$$

Differentiating the left hand side and using equation (2.2) we see that,

$$\begin{aligned}\frac{\partial}{\partial\Delta}\left\{\int \frac{d^4\ell}{(2\pi)^4}\left(\frac{\ell^2}{[\ell^2-\Delta]^3}-\frac{\ell^2}{[\ell^2-\Delta_\Lambda]^3}\right)\right\}&=3\int \frac{d^4\ell}{(2\pi)^4}\left(\frac{\ell^2}{[\ell^2-\Delta]^4}-\frac{\ell^2}{[\ell^2-\Delta_\Lambda]^4}\right),\\&=3\frac{i(-1)^3\cdot 2}{(4\pi)^2(3\cdot 2\cdot 1)}\left(\frac{1}{\Delta}-\frac{1}{\Delta_\Lambda}\right),\\&=\frac{i}{(4\pi)^2}\left(\frac{1}{\Delta_\Lambda}-\frac{1}{\Delta}\right),\\&=\frac{i}{(4\pi)^2}\left(\frac{\Delta-\Delta_\Lambda}{\Delta_\Lambda\Delta}\right),\\&=\frac{i}{(4\pi)^2}\frac{-z\Lambda^2}{\Delta_\Lambda\Delta},\end{aligned}$$

Therefore the derivatives of each sides of the desired identity with respect to Δ are equal. We note that, by direct calculation, the constant of integration is zero.

$$\therefore \int \frac{d^4\ell}{(2\pi)^4}\left(\frac{\ell^2}{[\ell^2-\Delta]^3}-\frac{\ell^2}{[\ell^2-\Delta_\Lambda]^3}\right)=\frac{i}{(4\pi)^2}\ln\left(\frac{\Delta_\Lambda}{\Delta}\right). \tag{2.4}$$

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3. The Volume Element in D-Dimensions

a) We note that evaluating the trivial Gaussian integral yields

$$I=\int_{-\infty}^{\infty}dx\; e^{-x^2}=\sqrt{\pi}. \tag{3.1}$$

b) Let us compute the general Gaussian integral,

$$I^n=\int_{-\infty}^{\infty}dx_1\cdots dx_n\; e^{-(x_1^2+\cdots+x_n^2)}.$$

We note that a general procedure for computing such Gaussian integrals is to convert it into an integral over spherical coordinates. Let us compute I^n directly this way. When needed, we will define the substitution variable $u\equiv r^2$.

$$\begin{aligned}I^n&=\int d\Omega_{n-1}\int_0^\infty dr\; r^{n-1}e^{-r^2},\\&=\int d\Omega_{n-1}\int_0^\infty \frac{du}{2r}\; r^{n-1}e^{-u},\\&=\int d\Omega_{n-1}\frac{1}{2}\int_0^\infty du\; u^{(n-2)/2}e^{-u},\\&=\frac{1}{2}\Gamma(n/2)\Omega_{n-1},\end{aligned}$$

c) Using our result above we see that $\pi^{(D/2)}=\Omega_{D-1}1/2\Gamma(D/2)$. Therefore it is clear that

$$\Omega_{D-1}=\frac{2\pi^{D/2}}{\Gamma(D/2)}. \tag{3.2}$$

d) Therefore by part (c) we see immediately that

$$\Omega_1=2\pi,\quad \Omega_2=4\pi,\quad \Omega_3=2\pi^2,\quad \Omega_4=\frac{8}{3}\pi^2. \tag{3.3}$$

4. The Electron Vertex Function

We are to completely simplify the numerator of the integrand of the electron vertex function's first order correction written as

$$\mathscr{N} \equiv \overline{u}(p') \left[\not{k}\gamma^\mu \not{k}' + m^2\gamma^\mu - 2m(k+k')^\mu \right] u(p). \tag{4.1}$$

In part to accomplish this task we will make the substitution

$$\ell \equiv k + yq - zp.$$

During the following exercise in algebra, we will often make use of the Dirac equation which can be written as

$$\overline{u}(p') \not{p}' = \overline{u}(p')m, \qquad \not{p}u(p) = mu(p), \qquad \overline{u}(p') \not{q}u(p) = 0, \tag{4.2}$$

and we will frequently imply the use of the Dirac equation to set $\not{p} \to m, \not{p}' \to m$, or $\not{q} \to 0$ by implying contraction with a spinor outside the square brackets. This of course can only be done when the specific momentum 4-vector is appropriately located (without γ^μ's between it and the needed spinor(s)). Also, we will make use of the facts derived in class that when this integral is evaluated, all terms linear in ℓ^ν will give no contribution and rotational symmetry allows us to set $\ell^\mu\ell^\mu \to \frac{1}{4}g^{\mu\nu}\ell^2$.[1]

Let us begin our calculation by direct substitution (making use of the stated identity to throw out terms linear in ℓ^ν).

$$\mathscr{N} = \overline{u}(p') \left[\underbrace{\not{\ell}\gamma^\mu \not{\ell}}_{\textbf{i}} - y(1-y)\underbrace{\not{q}\gamma^\mu \not{q}}_{\textbf{ii}} - zy\underbrace{\not{q}\gamma^\mu \not{p}}_{\textbf{iii}} + z(1-y)\underbrace{\not{p}\gamma^\mu \not{q}}_{\textbf{iv}} + z^2\underbrace{\not{p}\gamma^\mu \not{p}}_{\textbf{v}} + m^2\gamma^\mu - 2m(1-2y)q - 4mzp \right] u(p).$$

We will evaluate this in parts.

i. [2]

$$\not{\ell}\gamma^\mu \not{\ell} = 2\not{\ell}\ell - \not{\ell}^{\,2}\gamma^\mu = \frac{1}{2}g^{\mu\nu}\gamma_\nu\ell^2 - \ell^2\gamma^\mu = -\frac{1}{2}\ell^2\gamma^\mu.$$

ii.

$$\not{q}\gamma^\mu \not{q} = \underbrace{2\not{q}q}_{\to 0} - \not{q}^2\gamma^\mu = -q^2\gamma^\mu.$$

iii.

$$\not{q}\gamma^\mu \not{p} = \not{q}\gamma^\mu m = m\not{p}'\gamma^\mu - m\not{p}\gamma^\mu = m^2\gamma^\mu - 2mp^\mu + m^2\gamma^\mu = 2m^2\gamma^\mu - 2mp^\mu.$$

iv.

$$\not{p}\gamma^\mu \not{q} = \underbrace{2p\not{q}}_{\to 0} - \gamma^\mu \not{p}\not{q} = -2\gamma^\mu p\cdot q + m\gamma^\mu \not{q} = -\gamma^\mu 2p\cdot q + m\gamma^\mu \not{p}' - m^2\gamma^\mu,$$

$$= -\gamma^\mu 2p\cdot q + 2mp'^\mu - 2m^2\gamma^\mu.$$

Notice, however, that

$$2p\cdot q = p\cdot q + p\cdot q = p\cdot q + p'\cdot q - q^2 = p'^2 + p'\cdot p - p'\cdot p - p^2 - q^2 = m^2 - m^2 - q^2 = -q^2.$$

Therefore,

$$\not{p}\gamma^\mu \not{q} = \gamma^\mu q^2 + mp'^\mu - 2m^2\gamma^\mu.$$

v.

$$\not{p}\gamma^\mu \not{p} = m\not{p}\gamma^\mu = 2mp^\mu - m^2\gamma^\mu.$$

Combining all of these results, we may write the numerator as

$$\mathscr{N} = \overline{u}(p') \left[\gamma^\mu \overbrace{\left(-\frac{1}{2}\ell^2 + y(1-y)q^2 + z(1-y)q^2 - 2m^2yz - 2m^2z(1-y) - z^2m^2 + m^2 \right)}^{\mathscr{A}}, \right.$$

$$\left. + \underbrace{2myzp^\mu + 2mz(1-y)p'^\mu + 2mz^2p^\mu - 2m(1-2y)q^\mu - 4mzp^\mu}_{\mathscr{B}} \right] u(p).$$

[1]It is important to note that we do *not* imply that $\ell^\mu\ell^\nu = \frac{1}{4}g^{\mu\nu}\ell^2$ or that $\ell^\nu = 0$ but rather that these are symmetries of the integrand.

[2]Here and later in the derivation we make use of the identity $\not{p}^2 = p^2$. This is seen by simple γ algebra: $\not{p}^2 = p_\nu\gamma^\nu\gamma^\mu p_\mu = 2p^2 - p_\mu\gamma^\mu\gamma^\nu p_\nu = 2p^2 - \not{p}^2$. So $2\not{p}^2 = 2p^2 \Longrightarrow \not{p}^2 = p^2$.

Let us simplify the parts $\mathscr{A}$ and $\mathscr{B}$ separately. To do this, we will make repeated use of the fact that $x+y+z=1$ by the Dirac δ-functional of these Feynman parameters. Let us begin with part $\mathscr{A}$.

$$\begin{aligned}\mathscr{A} &= -\frac{1}{2}\ell^2 + q^2\left(y(1-y)+z(1-y)\right) + m^2\left(-2yz-2z(1-y)-z^2+1\right),\\ &= -\frac{1}{2}\ell^2 + q^2\left((1-x-z)(1-z)+z(1-y)\right) + m^2\left(-2yz-2z+2yz+1\right),\\ &= -\frac{1}{2}\ell^2 + q^2(1-x)(1-y) + m^2(1-2z-z^2).\end{aligned}$$

Now let us simplify part $\mathscr{B}$. This process will not seem beautiful or elegant, but in the words of Pascal, "I apologize for this [derivation's] length for I did not have time to make it short."

$$\begin{aligned}\mathscr{B} &= 2myzp^\mu + 2mz(1-y)p'^\mu + 2mz^2p^\mu - 2m(1-2y)q^\mu - 4mzp^\mu,\\ &= 2m\left(yzp^\mu + zp'^\mu - zyp'^\mu + z^2p^\mu - q^\mu + 2yq^\mu - 2zp^\mu\right),\\ &= 2m\left(z(z-1)p^\mu - zp^\mu + zp'^\mu - zp'^\mu + zxp'^\mu + z^2p'^\mu - q^\mu + 2yq^\mu + yzp^\mu\right),\\ &= 2m\left(z(z-1)(p^\mu+p'^\mu) + zq^\mu + zxp'^\mu - q^\mu + 2yq^\mu + yzp^\mu\right),\\ &= m\left(z(z-1)(p^\mu+p'^\mu) + z^2p^\mu - zp^\mu + z^2p'^\mu - zp'^\mu + 2zp'^\mu - 2zyp'^\mu - 2z^2p'^\mu\right.\\ &\qquad\qquad \left. + 2zp^\mu - 2xzp^\mu - 2z^2p'^\mu + 4yq^\mu + 2zq^\mu - 2q^\mu\right),\\ &= m\left(z(z-1)(p^\mu+p'^\mu) - z^2p^\mu + zp^\mu - z^2p'^\mu + zp'^\mu - 2zyp'^\mu - 2xzp^\mu + 4yq^\mu + 2zq^\mu - 2q^\mu\right),\\ &= m\left(z(z-1)(p^\mu+p'^\mu) - zp^\mu + zxp^\mu + zyp^\mu + zp^\mu - zp'^\mu + zxp'^\mu + ztp'^\mu + zp'^\mu - 2zyp'^\mu\right.\\ &\qquad\qquad \left. -2zxp^\mu + 4yp^\mu + 2zp'^\mu - 2zp^\mu - 2p'^\mu + 2p^\mu\right),\\ &= m\left(z(z-1)(p^\mu+p'^\mu) - zyp'^\mu + zxp'^\mu - zxp^\mu + zyp^\mu + 2yp'^\mu - 2yp^\mu + 2p'^\mu - 2xp'^\mu - 2zp'^\mu\right.\\ &\qquad\qquad \left. -2p^\mu + 2xp^\mu + 2zp^\mu + 2zp'^\mu - 2zp^\mu - 2p'^\mu + 2p^\mu\right),\\ &= m\left(z(z-1)(p^\mu+p'^\mu) + (p'^\mu - p^\mu)(zx - zy + 2y - 2x)\right),\\ &= mz(z-1)(p^\mu+p'^\mu) + mq^\mu(z-2)(x-y).\end{aligned}$$

When we combine these simplifications into the entire expression for the numerator, we see that

$$\therefore \mathscr{N} = \overline{u}(p')\left[\gamma^\mu\left(-\frac{1}{2}\ell^2 + (1-x)(1-y)q^2 + m^2(1-2z-z^2)\right) + mz(z-1)\left(p'^\mu + p^\mu\right) + m(z-2)(x-y)q^\mu\right].$$

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Physics 523, Quantum Field Theory II

Homework 3

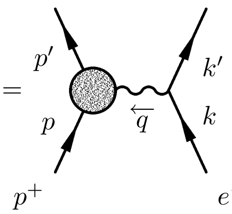
Due Wednesday, 28st January 2004

Jacob Lewis Bourjaily

The Rosenbluth Formula

We are to prove the *Rosenbluth Formula* by considering the elastic scattering of a relativistic electron off of a proton while correcting the vertex function of the proton. The amplitude for this process is,

$$i\mathscr{M} =$$



$$= \overline{u}(k')(-ie\gamma_\mu)u(k)\tfrac{-i}{q^2}\overline{u}(p')(-ie\Gamma^\mu)u(p).$$

a) Let us simplify the amplitude using the Gordon identity. Recall that we showed in class that the generalized vertex function Γ^μ may be written in terms of functions $F_1(q^2)$ and $F_2(q^2)$ as

$$\Gamma^\mu = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2).$$

Inserting this into the amplitude and recalling the Gordon identity, we see that

$$\begin{aligned}
i\mathscr{M} &= i\frac{e^2}{q^2}\overline{u}(k')\gamma_\mu u(k)\overline{u}(p')\Gamma^\mu u(p),\\
&= i\frac{e^2}{q^2}\overline{u}(k')\gamma_\mu u(k)\overline{u}(p')\left(\gamma^\mu F_1 + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2\right)u(p),\\
&= i\frac{e^2}{q^2}\overline{u}(k')\gamma_\mu u(k)\overline{u}(p')\left(\gamma^\mu F_1 + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2 + \frac{(p'+p)^\mu}{2m}F_2 - \frac{(p'+p)^\mu}{2m}F_2\right)u(p),\\
&= i\frac{e^2}{q^2}\overline{u}(k')\gamma_\mu u(k)\overline{u}(p')\left(\gamma^\mu (F_1+F_2) - \frac{(p'+p)^\mu}{2m}F_2\right)u(p),
\end{aligned}$$

$$\therefore \Gamma^\mu = \gamma^\mu(F_1+F_2) - \frac{(p'+p)^\mu}{2m}F_2.$$

b) Let us compute the spin-averaged amplitude squared directly. We see that

$$\begin{aligned}
\overline{|\mathscr{M}|^2} &= \frac{e^4}{4q^4}\sum_{\text{spin}}\overline{u}(k')\gamma_\mu u(k)\overline{u}(p')\left(\gamma^\mu(F_1+F_2) - \frac{(p'+p)^\mu}{2m}F_2\right)u(p)\overline{u}(p)\left(\gamma^\nu(F_1+F_2) - \frac{(p'+p)^\nu}{2m}F_2\right)u(p')\overline{u}(k)\gamma_\nu u(k'),\\
&= \frac{e^4}{4q^4}\mathrm{Tr}\left[(\not{k}'+m_e)\gamma_\mu(\not{k}+m_e)\gamma_\nu\right]\times\\
&\quad\left\{(F_1+F_2)^2\mathrm{Tr}\left[(\not{p}'+m)\gamma^\mu(\not{p}+m)\gamma^\nu\right] - F_2(F_1+F_2)\frac{(p'+p)^\nu}{2m}\mathrm{Tr}\left[(\not{p}'+m)\gamma^\mu(\not{p}+m)\right]\right.\\
&\quad\left.-F_2(F_1+F_2)\frac{(p'+p)^\mu}{2m}\mathrm{Tr}\left[(\not{p}'+m)(\not{p}+m)\gamma^\nu\right] + F_2^2\frac{(p'+p)^\mu(p'+p)^\nu}{4m^2}\mathrm{Tr}\left[(\not{p}'+m)(\not{p}+m)\right]\right\},\\
&= \frac{4e^4}{q^4}\left(k'_\mu k_\nu + k'_\nu k_\mu - g_{\mu\nu}(k'\cdot k - m_e^2)\right)\times\\
&\quad\left\{(F_1+F_2)^2\left(p'^\mu p^\nu + p'^\nu p^\mu - g^{\mu\nu}(p'\cdot p - m^2)\right) - F_2(F_1+F_2)(p'+p)^\mu(p'+p)^\nu\right.\\
&\quad\left.+\frac{F_2^2}{4m^2}(p'+p)^\mu(p'+p)^\nu(p'\cdot p + m^2)\right\},\\
&= \frac{4e^4}{q^4}\left(k'_\mu k_\nu + k'_\nu k_\mu - g_{\mu\nu}(k'\cdot k - m_e^2)\right)\times\\
&\quad\left\{(F_1+F_2)^2\left(p'^\mu p^\nu + p'^\nu p^\mu - g^{\mu\nu}(p'\cdot p - m^2)\right) + (p'+p)^\mu(p'+p)^\nu\left(\frac{p'\cdot p + m^2}{4m^2}F_2^2 - F_2(F_1+F_2)\right)\right\},
\end{aligned}$$

$$\begin{aligned}
\therefore \overline{|\mathscr{M}|^2} = \frac{8e^4}{q^4}&\left[(F_1+F_2)^2\left(k'\cdot p' k\cdot p + k'\cdot p k\cdot p - k'\cdot k m^2 - p'\cdot p m_e^2 + 2m^2m_e^2\right)\right.\\
&\left.+\left(\frac{p'\cdot p + m^2}{4m^2}F_2^2 - F_2(F_1+F_2)\right)\left(k'\cdot(p'+p)k\cdot(p'+p) - \frac{1}{2}(k'\cdot k - m_e^2)(p'+p)^2\right)\right].
\end{aligned}$$

c) Let us consider the kinematics of this reaction in the initial rest frame of the proton. In this frame we see that $p = (m, \vec{0}), k = (E, E\hat{z}), k' = (E', \vec{k'}), p' = (E - E' + m, -\vec{k})$ with $|\vec{k'}| = E'$. We have defined the momentum transfer q such that $p' - p = q = k - k'$.

Noting that $p \cdot p' = m^2 + Em - E'm$, let us compute p'^2.

$$p'^2 = (p+q)^2 = p^2+2p\cdot q+q^2 = m^2+2p\cdot(p'-p)+q^2 = -m^2+2p'\cdot p+q^2 = m^2+2Em-2E'm+q^2 = m^2,$$
$$\implies q^2 = 2E'm - 2Em,$$
$$\therefore E' = E + \frac{q^2}{2m}.$$

If we write $k' = (E', 0, E'\sin\theta, \cos\theta)$ so that $q = (E - E', 0, -E'\sin\theta, E - E'\cos\theta)$ we see

$$q^2 = E'^2 - 2EE' + E^2 - E'^2 - E'^2\sin^2\theta - E^2 + 2EE'\cos\theta - E'^2\cos^2\theta = 2EE'(\cos\theta - 1) = -4EE'\sin^2\frac{\theta}{2}.$$

Using our identity derived above that $E' = E + \frac{q^2}{2m}$, we may conclude that

$$q^2 = -4E^2\sin^2\frac{\theta}{2} - \frac{q^2}{2m}4E\sin^2\frac{\theta}{2},$$
$$\therefore q^2 = -\frac{4E^2\sin^2\frac{\theta}{2}}{1 + \frac{2E}{m}\sin^2\frac{\theta}{2}}.$$

Let us now compute all of the required inner products to compute the desired amplitude squared. Noting that $p^2 = p'^2 = m^2, k^2 = k'^2 = 0$, and $p \cdot k = Em$ we may derive all of our necessary identities and inner products indirectly (it's more fun that way). We notice that

$$p'^2 = m^2 = p^2 + 2p\cdot q + q^2 = -m^2 + 2p'\cdot p + q^2,$$
$$\therefore p'\cdot p = m^2 - \frac{q^2}{2}.$$

Similarly,

$$p'^2 = m^2 = p^2 + 2p\cdot q + q^2 = m^2 + 2p\cdot k - 2p\cdot k' + q^2,$$

but we know that $p \cdot k = Em$,

$$\therefore p\cdot k' = Em + \frac{q^2}{2}.$$

Likewise,

$$k'^2 = 0 = k^2 - 2k\cdot q + q^2 = 2k\cdot k' + q^2 = 0,$$
$$\therefore k'\cdot k = -\frac{q^2}{2}.$$

And

$$k'^2 = 0 = k^2 - 2k\cdot q + q^2 = -2k\cdot p' + 2k\cdot p + q^2,$$

where we know that $k \cdot p = Em$ and

$$\therefore k\cdot p' = Em + \frac{q^2}{2}.$$

Similarly,

$$k^2 = 0 = k'^2 + 2q\cdot k' + q^2 = 2p'\cdot k' + q^2,$$
$$\therefore p'\cdot k' = Em.$$

Tabulating our results, we have shown that

$$k'\cdot k = -\frac{q^2}{2} \qquad p'\cdot p = m^2 - \frac{q^2}{2} \qquad k'\cdot p = Em + \frac{q^2}{2}$$
$$p'\cdot k = Em + \frac{q^2}{2} \qquad k'\cdot p' = Em \qquad p\cdot k = Em.$$

These imply that

$$k\cdot(p'+p) = 2Em + \frac{q^2}{2}, \quad k'\cdot(p'+p) = 2Em + \frac{q^2}{2}, \quad \text{and} \quad (p+p')^2 = 4m^2 - q^2.$$

d) We are to use the kinematic information derived in part (c) above to rewrite the spin-averaged amplitude squared into a more convenient form. Recall that

$$\overline{|\mathscr{M}|^2} = \frac{8e^4}{q^4}\Bigg[(F_1+F_2)^2 \overbrace{\left(k'\cdot p' k\cdot p + k'\cdot p k\cdot p - k'\cdot k m^2 - p'\cdot p m_e^2 + 2m^2 m_e^2\right)}^{\text{i}} + \underbrace{\left(\frac{p'\cdot p + m^2}{4m^2}F_2^2 - F_2(F_1+F_2)\right)}_{\text{ii}}\underbrace{\left(k'\cdot(p'+p)k\cdot(p'+p) - \frac{1}{2}(k'\cdot k - m_e^2)(p'+p)^2\right)}_{\text{iii}}\Bigg].$$

We note that in the approximation where $k^2 \sim 0$, we should set $m_e \to 0$. Let us compute each part separately first before combining the results.

i. $$(k'\cdot p')(k\cdot p) + (k'\cdot p)(k\cdot p') - (k'\cdot k)m^2 = (Em)^2 + (Em)^2 + Emq^2 + \frac{q^4}{4} + \frac{q^2}{2}m^2.$$

ii.
$$\begin{aligned}\frac{p'\cdot p + m^2}{4m^2}F_s^2 - F_1F_2 - F_2^2 &= \frac{1}{2}F_2^2 - \frac{q^2}{8m^2}F_2^2 - F_1F_2 - F_2^2,\\ &= -\frac{1}{2}\left[(F_2^2 + 2F_1F_2 + F_1^2 - F_1^2 + \frac{q^2}{4m^2}F_2^2\right],\\ &= -\frac{1}{2}\left[((F_1+F_2)^2 - \left(F_1^2 - \frac{q^2}{4m^2}F_2^2\right)\right].\end{aligned}$$

iii. $$(k'\cdot(p'+p))(k\cdot(p'+p)) - \frac{1}{2}(k'\cdot k)(p'+p)^2 = 4(Em)^2 + 2Emq^2 + \frac{q^4}{4} + q^2m^2 - \frac{q^4}{4}.$$

Combining these results, we see that the coefficient for the $(F_1+F_2)^2$ term will be

$$2(Em)^2 + Emq^2 + \frac{q^4}{4} + \frac{q^2}{2}m^2 - 2(Em)^2 - Emq^2 - \frac{q^2}{2}m^2 = \frac{q^4}{4},$$

which can be written,

$$\frac{q^4}{4} = \frac{q^2}{2}\frac{q^2}{2} = -\frac{2E^2m^2}{1+\frac{2E}{m}\sin^2\frac{\theta}{2}}\frac{q^2}{2m^2}.$$

Similarly, we will combine the results above to compute the coefficient for the $(F_1^2 - \frac{q^2}{4m^2}F_2^2)$ term.

$$\begin{aligned}2(Em)^2 + Emq^2 + \frac{q^2}{2}m^2 &= 2E^2m^2 - \frac{4E^3m\sin^2\frac{\theta}{2}}{1+\frac{2E}{m}\sin^2\frac{\theta}{2}} - \frac{2E^2m^2\sin^2\frac{\theta}{2}}{1+\frac{2E}{m}\sin^2\frac{\theta}{2}},\\ &= \frac{2E^2m^2 + 4E^3m\sin^2\frac{\theta}{2} - 4E^3m\sin^2\frac{\theta}{2} - 2m^2E^2\sin^2\frac{\theta}{2}}{1+\frac{2E}{m}\sin^2\frac{\theta}{2}},\\ &= 2E^2m^2\frac{1-\sin^2\frac{\theta}{2}}{1+\frac{2E}{m}\sin^2\frac{\theta}{2}} = \frac{2E^2m^2\cos^2\frac{\theta}{2}}{1+\frac{2E}{m}\sin^2\frac{\theta}{2}}.\end{aligned}$$

Therefore, combining all of these results, the total spin-average amplitude squared becomes

$$\overline{|\mathscr{M}|^2} = \frac{16e^4E^2m^2}{q^4\left(1+\frac{2E}{m}\sin^2\frac{\theta}{2}\right)}\left[\left(F_1^2 - \frac{q^2}{4m^2}F_2^2\right)\cos^2\frac{\theta}{2} - \frac{q^2}{2m^2}(F_1+F_2)^2\sin^2\frac{\theta}{2}\right].$$

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e) Let us compute the differential cross section, $\frac{d\sigma}{d\cos\theta}\big|_{\text{lab}}$. To do this, we will compute the cross section in most general terms. From elementary considerations, we calculated that

$$d\sigma = \frac{1}{2E_{\mathcal{A}}2E_{\mathcal{B}}|v_{\mathcal{A}} - v_{\mathcal{B}}|}\left(\prod_f \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\right)\overline{|\mathscr{M}|^2}(2\pi)^4\delta^{(4)}\left(p_{\mathcal{A}} + p_{\mathcal{B}} - \sum p_f\right),$$
$$= \frac{1}{4mE}\overline{|\mathscr{M}|^2}\frac{d^3p'd^3k'}{(2\pi)^2}\frac{1}{4E'E'_p}\delta^{(4)}(p + k - p' - k').$$

We see that this is so because $E_{\mathcal{A}} = m$, $E_{\mathcal{B}} = E$, $|v_{\mathcal{A}} - v_{\mathcal{B}}| = 1$ and there are two final states.

Let us now integrate over $d\sigma$ to find its dependence on $\cos\theta$. During the derivation, we will make use of the fact that $E+m = E'_p+E'$ by energy conservation enforced by the dirac δ function. We will also call upon our results above to use the identities $E' = E + \frac{q^2}{2m}$ and $q^2 = \frac{4E^2\sin^2\frac{\theta}{2}}{1+\frac{2E}{m}\sin^2\frac{\theta}{2}}$. Notice the insertion of the Jacobian for the change of variables to integrate over the energy portion of the δ function in line 4. We will now proceed directly by first integrating over the p' part of the integral.

$$\sigma = \int d\sigma = \frac{1}{4mE}\overline{|\mathscr{M}|^2}\int\frac{d^3p'd^3k'}{(2\pi)^2}\frac{1}{4E'E'_p}\delta^{(4)}(p + k - p' - k'),$$
$$= \frac{1}{4mE}\overline{|\mathscr{M}|^2}\int\frac{d^3k'}{(2\pi)^2}\frac{1}{4E'E'_p}\delta^{(1)}\left(E' - E - m + \sqrt{m^2 + E^2 + E'^2 - 2EE'\cos\theta}\right),$$
$$= \frac{1}{4mE}\overline{|\mathscr{M}|^2}\int\frac{E'^2dEd\Omega}{(2\pi)^2}\frac{1}{4E'E'_p}\delta^{(1)}\left(E' - E - m + \sqrt{m^2 + E^2 + E'^2 - 2EE'\cos\theta}\right),$$
$$= \frac{1}{4mE}\overline{|\mathscr{M}|^2}\int\frac{d\Omega}{(2\pi)^2}\frac{E'}{4E'_p}\left(1 + \frac{E' - E\cos\theta}{E'_p}\right)^{-1},$$
$$= \frac{1}{4mE}\overline{|\mathscr{M}|^2}\int\frac{d\Omega}{(2\pi)^2}\frac{E'}{4E'_p}\left(\frac{E'_p}{E'_p + E' - E\cos\theta}\right),$$
$$= \frac{1}{4mE}\overline{|\mathscr{M}|^2}\int\frac{d\cos\theta}{(2\pi)}\frac{E'}{4E'_p}\left(\frac{E'_p}{E'_p + E' - E\cos\theta}\right),$$
$$= \frac{1}{32\pi mE}\overline{|\mathscr{M}|^2}\int d\cos\theta\frac{E'}{m + E(1 - \cos\theta)},$$
$$= \frac{1}{32\pi m^2E}\overline{|\mathscr{M}|^2}\int d\cos\theta\frac{E'}{1 + \frac{2E}{m}\sin^2\frac{\theta}{2}},$$
$$= \frac{1}{32\pi m^2E}\overline{|\mathscr{M}|^2}\int d\cos\theta\frac{E + \frac{q^2}{2m}}{1 + \frac{2E}{m}\sin^2\frac{\theta}{2}},$$
$$= \frac{1}{32\pi m^2E}\overline{|\mathscr{M}|^2}\int d\cos\theta\frac{E - \frac{2E^2\sin^2\frac{\theta}{2}}{m(1+\frac{2E}{m}\sin^2\frac{\theta}{2})}}{1 + \frac{2E}{m}\sin^2\frac{\theta}{2}},$$
$$= \frac{1}{32\pi m^2E}\overline{|\mathscr{M}|^2}\int d\cos\theta\frac{E + \frac{2E^2}{m}\sin^2\frac{\theta}{2} - \frac{2E^2}{m}\sin^2\frac{\theta}{2}}{(1 + \frac{2E}{m}\sin^2\frac{\theta}{2})^2},$$
$$= \frac{1}{32\pi m^2E}\overline{|\mathscr{M}|^2}\int d\cos\theta\frac{E}{(1 + \frac{2E}{m}\sin^2\frac{\theta}{2})^2},$$
$$= \frac{1}{32\pi m^2(1 + \frac{2E}{m}\sin^2\frac{\theta}{2})^2}\overline{|\mathscr{M}|^2}\int d\cos\theta,$$
$$= \frac{1}{32\pi m^2(1 + \frac{2E}{m}\sin^2\frac{\theta}{2})^2}\overline{|\mathscr{M}|^2}\cos\theta,$$

$$\therefore \left.\frac{d\sigma}{d\cos\theta}\right|_{\text{lab}} = \frac{1}{32\pi m^2\left(1 + \frac{2E}{m}\sin^2\frac{\theta}{2}\right)^2}\overline{|\mathscr{M}|^2}.$$

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f) We will now derive the Rosenbluth formula. From our work above, we see that

$$\begin{aligned}
\left.\frac{d\sigma}{d\cos\theta}\right|_{\text{lab}} &= \frac{16e^4E^2m^2\left[\left(F_1^2-\frac{q^2}{4m^2}F_2^2\right)\cos^2\frac{\theta}{2}-\frac{q^2}{2m^2}(F_1+F_2)^2\sin^2\frac{\theta}{2}\right]}{32\pi m^2\left(1+\frac{2E}{m}\sin^2\frac{\theta}{2}\right)^2q^4\left(1+\frac{2E}{m}\sin^2\frac{\theta}{2}\right)},\\
&= \frac{e^4E^2\left[\left(F_1^2-\frac{q^2}{4m^2}F_2^2\right)\cos^2\frac{\theta}{2}-\frac{q^2}{2m^2}(F_1+F_2)^2\sin^2\frac{\theta}{2}\right]}{2\pi\frac{16E^4\sin^4\frac{\theta}{2}}{\left(1+\frac{2E}{m}\sin^2\frac{\theta}{2}\right)^2}\left(1+\frac{2E}{m}\sin^2\frac{\theta}{2}\right)^2\left(1+\frac{2E}{m}\sin^2\frac{\theta}{2}\right)},\\
&= \frac{e^4\left[\left(F_1^2-\frac{q^2}{4m^2}F_2^2\right)\cos^2\frac{\theta}{2}-\frac{q^2}{2m^2}(F_1+F_2)^2\sin^2\frac{\theta}{2}\right]}{32\pi E^2\sin^4\frac{\theta}{2}\left(1+\frac{2E}{m}\sin^2\frac{\theta}{2}\right)},
\end{aligned}$$

$$\therefore\ \left.\frac{d\sigma}{d\cos\theta}\right|_{\text{lab}} = \frac{\pi\alpha^2\left[\left(F_1^2-\frac{q^2}{4m^2}F_2^2\right)\cos^2\frac{\theta}{2}-\frac{q^2}{2m^2}(F_1+F_2)^2\sin^2\frac{\theta}{2}\right]}{2E^2\sin^4\frac{\theta}{2}\left(1+\frac{2E}{m}\sin^2\frac{\theta}{2}\right)}.$$

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Physics 523, Quantum Field Theory II

Homework 4

Due Wednesday, 4th February 2004

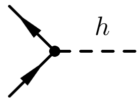
Jacob Lewis Bourjaily

The Anomalous Magnetic Moments of e^- and μ^-

We are to investigate the possible contributions of scalar loops to the QED anomalous magnetic moments of the electron and muon. First we will consider contributions from a Higgs particle, h. We casually note that because the interaction Hamiltonian is given by,

$$H_{\text{int}} = \int d^x \; \frac{\lambda}{\sqrt{2}} h\overline{\psi}\psi,$$

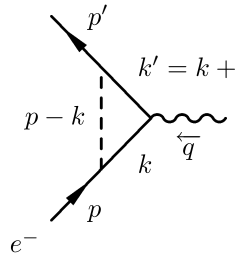
our vertex rule is



$$= -i\frac{\lambda}{\sqrt{2}}.$$

Therefore, we may now compute the amplitude for the following interaction.

$i\mathscr{M} =$



$$= \int \frac{d^4k}{(2\pi)^4} \overline{u}(p') \frac{-i\lambda}{\sqrt{2}} \frac{i}{((p-k)^2 - m_h^2 + i\epsilon)} \frac{i(\not{k}' + m)}{(k'^2 - m^2 + i\epsilon)} (-ie\gamma^\mu) \frac{i(\not{k} + m)}{(k^2 - m^2 + i\epsilon)} \frac{-i\lambda}{\sqrt{2}} u(p),$$

$$\boxed{\therefore i\mathscr{M} = \frac{e\lambda^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{\overline{u}(p')\left[(\not{k}' + m)\gamma^\mu(\not{k} + m)\right] u(p)}{(k^2 - m^2 + i\epsilon)(k'^2 - m^2 + i\epsilon)((p-k)^2 - m_h^2 + i\epsilon)}.} \quad \text{(a.2)}$$

Let us now simplify the denominator using Feynman parametrization. Using the same procedure as before, we see that we may reduce the denominator to the form,

$$\frac{1}{(k^2 - m_e^2 + i\epsilon)(k'^2 - m_e^2 + i\epsilon)((p-k)^2 - m_h^2 + i\epsilon)},$$

$$= \int dxdydz\delta^{(3)}(x + y + z - 1) \frac{2}{[xk^2 + yk^2 + zk^2 + 2yqk + yq^2 + zp^2 - 2zpk - xm^2 - ym^2 - zm_h^2 + (x + y + z)i\epsilon]^3},$$

$$= \int dxdydz\delta^{(3)}(x + y + z - 1) \frac{2}{[k^2 + 2k(yq - zp) + yq^2 + zp^2 - (1 - z)m^2 - zm_h^2 + i\epsilon]^3},$$

Introducing the terms,

$$\ell \equiv k + yq - zp \qquad \text{and} \qquad \Delta = -xyq^2 + (1 - z)^2m^2 + zm_h^2,$$

we see that the denominator becomes,

$$\boxed{\int dxdydz\delta^{(3)}(x + y + z - 1) \frac{2}{[\ell^2 - \Delta + i\epsilon]^3}} \quad \text{(a.3)}$$

We are now ready to simplify the numerator of the integrand using the parameters ℓ for equation (a.2) above. There are arguably more elegant ways to go about this calculation, but we will simplify by brute force. We will use, without repeated demonstration, several identities that were shown in homework 2. Specifically, we will expand the integrand with the knowledge that all terms linear in ℓ will integrate to zero and so may be ignored. Furthermore, we are only interested in terms that do not involve a γ^μ so in the below tabulation of results from the Dirac algebra, we will simply write $\not{q}\gamma^\mu \to -2p^\mu$ with knowledge that $\not{q}\gamma^\mu = 2m\gamma^\mu - 2p^\mu$ because we are uninterested in terms proportional to γ^μ.

We will begin our simplification with a full expansion of the numerator as follows:

$$\begin{aligned}\mathscr{N} &= \overline{u}(p')\left[(\not{k}' + m)\gamma^\mu(\not{k} + m)\right],\\ &= \overline{u}(p')\left[\not{k}'\gamma^\mu\not{k} + m\not{k}'\gamma^\mu + m\gamma^\mu\not{k} + m^2\gamma^\mu\right]u(p),\\ &\to \overline{u}(p')\Big[\underbrace{\not{\ell}\gamma^\mu\not{\ell}}_{\mathbf{i}} - y(1-y)\underbrace{\not{q}\gamma^\mu\not{q}}_{\mathbf{ii}} + z(1-y)\underbrace{\not{q}\gamma^\mu\not{p}}_{\mathbf{iii}} - zy\underbrace{\not{p}\gamma^\mu\not{q}}_{\mathbf{iv}} + z^2\underbrace{\not{p}\gamma^\mu\not{p}}_{\mathbf{v}} + m^2\gamma^\mu\\ &\qquad + m(1-y)\underbrace{\not{q}\gamma^\mu}_{\mathbf{vi}} + mz\underbrace{\not{p}\gamma^\mu}_{\mathbf{vii}} - my\underbrace{\gamma^\mu\not{q}}_{\mathbf{viii}} + mz\underbrace{\gamma^\mu\not{p}}_{\mathbf{ix}}\Big]u(p).\end{aligned}$$

Using Dirac algebra and our results from homework 2, we see that

$$\begin{array}{lll}(\mathbf{i}) \to 0, & (\mathbf{ii}) \to 0, & (\mathbf{iii}) \to -2mp^\mu,\\ (\mathbf{iv}) \to 2mp^\mu, & (\mathbf{v}) \to 2mp^\mu, & (\mathbf{vi}) \to -2p^\mu,\\ (\mathbf{vii}) \to 2p^\mu, & (\mathbf{viii}) \to 2p'^\mu, & (\mathbf{xi}) \to 0.\end{array}$$

Using this result (which ignores all terms linear in ℓ and γ^μ), we see that

$$\begin{aligned}\mathscr{N} &\to \overline{u}(p')\Big[-2mz(1-y)p^\mu - 2mzyp'^\mu + 2mz^2p^\mu - 2m(1-y)p^\mu + 2mzp^\mu - 2myp'^\mu\Big]u(p),\\ &= \overline{u}(p')\Big[mp'^\mu(-2zy - 2y) + mp^\mu(2zy + 2y + 2z^2 - 2)\Big]u(p),\\ &= \overline{u}(p')\Big[m(p'^\mu - p^\mu)(-2zy - 2y) + mp^\mu(2z^2 - 2)\Big]u(p),\\ &= \overline{u}(p')\Big[m(p'^\mu - p^\mu)(-2zy - 2y) + mp^\mu(2z^2 - 2) + mp'^\mu(z^2 - 1) - mp'^\mu(z^2 - 1)\Big]u(p),\\ &= \overline{u}(p')\Big[(p'^\mu + p^\mu)m(z^2 - 1) + (p'^\mu - p^\mu)m(1 - z^2 - 2zy - 2y)\Big]u(p),\end{aligned}$$

$$\boxed{\therefore \mathscr{N} \to \overline{u}(p')\Big[(p'^\mu + p^\mu)m(z^2 - 1) + (p'^\mu - p^\mu)m(y - x)(z - 1)\Big]u(p).} \tag{a.4}$$

We notice almost trivially that this satisfies the Ward identity because the term proportional to $q^\mu = (p'^\mu - p^\mu)$ is odd under the interchange of $x \leftrightarrow y$ while the integral is symmetric under $x \leftrightarrow y$. Therefore the term proportional q^μ will vanish when integrated.

Recall that our goal is to discover this diagram's contribution to the anomalous magnetic moment, the $F_2(q^2)$ term. We recall that we have defined the corrected vertex function Γ^μ in terms of the functions F_1 and F_2 as

$$\Gamma^\mu = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2).$$

Because the term proportional to $(p'^\mu + p^\mu)$ is multiplied on the outside by $\overline{u}(p')$ and $u(p)$, we may use the Gordon identity to express it in terms of $\frac{i\sigma^{\mu\nu}q_\nu}{2m}$ and γ^μ. Because we are generally ignoring all terms proportional to γ^μ, we may substitute

$$m(z^2 - 1)(p'^\mu + p) \to 2m^2(1 - z^2)\frac{i\sigma^{\mu\nu}q_\nu}{2m}.$$

Because $F_2(q^2)$ is the term proportional to the $\frac{i\sigma^{\mu\nu}q_\nu}{2m}$ term, we see that this implies that

$$F_2(q^2) = \int dxdydz\delta^{(3)}(x + y + z - 1)\int \frac{d^4\ell}{(2\pi)^4}\frac{i\lambda^2}{2}\frac{2m^2(1 - z^2)2}{[\ell^2 - \Delta + i\epsilon]^3}.$$

We may simplify this integral substantially by recalling our work in homework 2 when we computed general integrals of this form. Taking the limit of $q \to 0$, we see that

$$\begin{aligned}
F_2(q^2) &= \int dxdydz\delta^{(3)}(x+y+z-1)\int \frac{d^4\ell}{(2\pi)^4}\frac{i\lambda^2}{2}\frac{2m^2(1-z^2)2}{[\ell^2-\Delta+i\epsilon]^3},\\
&= \int dxdydz\delta^{(3)}(x+y+z-1)\left[\frac{i\lambda^2}{2}\frac{-i}{(4\pi)^2}\frac{4m^2(1-z^2)}{2}\frac{1}{\Delta}\right],\\
&= \frac{\lambda^2 m_e^2}{16\pi^2}\int dxdydz\delta^{(3)}(x+y+z-1)\frac{(1-z^2)}{zm_h^2+(1-z)^2m_e^2},\\
&= \frac{\lambda^2 m_e^2}{16\pi^2}\int_0^1 dz\frac{(1-z)(1-z^2)}{zm_h^2+(1-z)^2m_e^2},\\
&\approx \frac{\lambda^2 m_e^2}{16\pi^2}\left[\int_0^1 dz\frac{1}{zm_h^2+(1-z)^2m_e^2}-\frac{1}{m_h^2}\int_0^1 dz 1+z-z^2\right],\\
&= \frac{\lambda^2 m_e^2}{16\pi^2 m_h^2}\left[\int_0^1 dz\frac{1}{z+(1-z)^2\frac{m_e^2}{m_h^2}}-\frac{7}{6}\right]. \qquad \text{(a.8)}
\end{aligned}$$

Now, let us simplify this formula in the limit where the Higgs mass is very much larger than the electron.

$$\begin{aligned}
F_2(q^2) &\approx \frac{\lambda^2 m_e^2}{16\pi^2 m_h^2}\left[\frac{1}{1-\frac{m_e^2}{m_h^2}}\int_{\frac{m_e^2}{m_h^2}}^1 du\frac{1}{u}-\frac{7}{6}\right],\\
&= \frac{\lambda^2 m_e^2}{16\pi^2 m_h^2}\left[\frac{1}{1-\frac{m_e^2}{m_h^2}}\left(\ln(1)-\ln\left(\frac{m_e^2}{m_h^2}\right)\right)-\frac{7}{6}\right],
\end{aligned}$$

$$\boxed{\therefore F_2(q^2) \approx \frac{\lambda^2 m_e^2}{16\pi^2 m_h^2}\left[\ln\left(\frac{m_h^2}{m_e^2}\right)-\frac{7}{6}\right].} \qquad \text{(b.1)}$$

Let us try to compute this contribution for real experimental numbers. We can take a more or less 'good' estimate of the Higgs vacuum expectation value as $v = 246\text{GeV}$. We know that the coupling constant λ may be written in terms of the experimental mass of the electron as $\lambda_e = \frac{m_e}{v}\sqrt{2} \approx 2.94 \times 10^{-6}$. If we take a rather hopeful estimate for the Higgs mass, we can assume it is near its lower experimental bound at $m_h \approx 114\text{GeV}$. Using these numbers, we calculate an anomalous magnetic moment contribution of

$$\boxed{\delta_{\text{higgs}} a_e \approx 2.58 \times 10^{-23}.} \qquad \text{(b.2)}$$

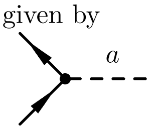
For the muon, we get a coupling to the Higgs of $\lambda_\mu = \frac{m_\mu}{v}\sqrt{2} \approx 6.03 \times 10^{-4}$. Using the same approximate Higgs mass of 114GeV, we see that the anomalous magnetic moment of the muon is altered by

$$\boxed{\delta_{\text{higgs}} a_\mu \approx 2.51 \times 10^{-14}.} \qquad \text{(b.3)}$$

Let us now consider the contribution given for an interaction with an axion particle given by the interaction Hamiltonian

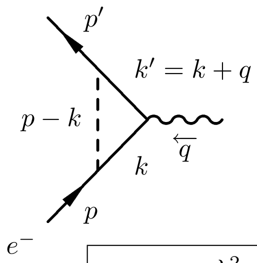
$$H = \int d^x \frac{i\lambda}{\sqrt{2}} a\overline{\psi}\gamma^5\psi.$$

We see immediately that our vertex rule is given by



$$= \frac{\lambda}{\sqrt{2}}\gamma^5.$$

Let us now write out the amplitude for the axion's contribution to the vertex function. We see that

$i\mathscr{M} =$ 

$$= \int \tfrac{d^4k}{(2\pi)^4}\overline{u}(p')\tfrac{\lambda}{\sqrt{2}}\gamma^5 \tfrac{i}{((p-k)^2-m_a^2+i\epsilon)}\tfrac{i(\not{k}'+m)}{(k'^2-m^2+i\epsilon)}(-ie\gamma^\mu)\tfrac{i(\not{k}+m)}{(k^2-m^2+i\epsilon)}\gamma^5\tfrac{\lambda}{\sqrt{2}}u(p),$$

$$\boxed{\therefore i\mathscr{M} = \frac{e\lambda^2}{2}\int\frac{d^4k}{(2\pi)^4}\frac{-\overline{u}(p')\left[\gamma^5(\not{k}'+m)\gamma^\mu(\not{k}+m)\gamma^5\right]u(p)}{(k^2-m^2+i\epsilon)(k'^2-m^2+i\epsilon)((p-k)^2-m_a^2+i\epsilon)}.} \tag{c.1}$$

We can simplify the numerator and demoninator as before. Notice that the only change in the denominator algebra is that $\Delta = -xyq^2 + (1-z)^2m_e^2 - zm_a^2$. In the numerator, we can commute the γ^5 through each of the terms to get a minus sign relative to the 'slash' terms. When we also take into account the overall minus which multiplies the numerator, we arrive at

$$i\mathscr{M} = \frac{e\lambda^2}{2}\int\frac{d^4k}{(2\pi)^4}\frac{\overline{u}(p')\left[(\not{k}'-m)\gamma^\mu(\not{k}-m)\right]u(p)}{(k^2-m^2+i\epsilon)(k'^2-m^2+i\epsilon)((p-k)^2-m_a^2+i\epsilon)}.$$

This is of course very similar to the equation derived in parts (a). Recall when we expanded all of the terms for the Higgs, we had some of the 'm' terms that came from the Dirac algebra and some explicit the equation as above. Taking these differences into account, we can use our work from part (a) to arrive at a simplified numerator.

$$\begin{aligned}
\mathscr{N} &\to \overline{u}(p')\Big[-2mz(1-y)p^\mu - 2mzyp'^\mu + 2mz^2p^\mu + 2m(1-y)p^\mu - 2mzp^\mu + 2myp'^\mu\Big]u(p),\\
&= \overline{u}(p')\Big[mp^\mu(-2z(1-y)+2z^2+2-2y-2z) + mp'^\mu(-2zy+2y)\Big]u(p),\\
&= \overline{u}(p')\Big[m(p'^\mu - p^\mu)(2y-2zy)m + mp^\mu(-4z+2z^2+2)\Big]u(p),\\
&= \overline{u}(p')\Big[m(p'^\mu - p^\mu)(2y-2zy)m + mp^\mu(-4z+2z^2+2) + mp'^\mu(1-z)^2 - mp'^\mu(1-z)^2\Big]u(p),\\
&= \overline{u}(p')\Big[(p'^\mu + p^\mu)(1-z)^2m + (p'^\mu - p^\mu)(2y-2zy-(1-z)^2)m\Big]u(p).
\end{aligned}$$

Again, using the Gordong identity, we may write the contribution to $F_2(q^2)$ as

$$\begin{aligned}
F_2(q^2) &= \int dxdydz\delta^{(3)}(x+y+z-1)\int\frac{d^4\ell}{(2\pi)^4}\frac{i\lambda^2}{2}\frac{2m^2(1-z)^2 2}{[\ell^2-\Delta+i\epsilon]^3},\\
&= \int dxdydz\delta^{(3)}(x+y+z-1)\left[\frac{i\lambda^2}{2}\frac{-i}{(4\pi)^2}\frac{4m^2(1-z)^2}{2}\frac{1}{\Delta}\right],
\end{aligned}$$

$$\boxed{\therefore F_2(q^2) = \frac{\lambda^2m_e^2}{16\pi^2}\int_0^1 dz\frac{(1-z)^3}{zm_a^2+(1-z)^2m_e^2}.} \tag{c.2}$$

Now, this integral cannot be so easily takn in the limit of a heavy axion. In fact, experimental evidence strongly limits the mass of the axion to be very, very light. The most restrictive data, from Supernova 1987a, restricts $m_a \lesssim 10^{-5}$eV. In the limit where the axion is very, very much lighter than the electron, we see that

$$\begin{aligned}
F_2(q^2) &= \frac{\lambda^2m_e^2}{16\pi^2}\int_0^1 dz\frac{(1-z)^3}{zm_a^2+(1-z)^2m_e^2},\\
&\approx \frac{\lambda^2}{16\pi^2}\int_0^1 dz\frac{(1-z)^3}{(1-z)^2} = \frac{\lambda^2}{32\pi^2},
\end{aligned}$$

$$\boxed{\therefore \delta_{\text{axion}}a_e \approx \delta_{\text{axion}}a_\mu \approx \frac{\lambda^2}{32\pi^2}.} \tag{c.3}$$

Physics 523, Quantum Field Theory II

Homework 5

Due Wednesday, 11th February 2004

Jacob Lewis Bourjaily

The Electron Self-Energy

1. We are to verify the equation,

$$\int \frac{d^4\ell}{(2\pi)^4}\left(\frac{1}{[\ell^2-\Delta]^2}-\frac{1}{[\ell^2-\Delta_\Lambda]^2}\right)=\frac{i}{(4\pi)^2}\log\left(\frac{\Delta_\Lambda}{\Delta}\right).$$

To evaluate this, we will consider differentiation of the integral with respect to both Δ and Δ_Λ, considering them as separate, independent variables. Because the integration will commute with these derivatives, we may use our results of to see

$$\begin{aligned}
\frac{d}{d\Delta}\frac{d}{d\Delta_\Lambda}\int \frac{d^4\ell}{(2\pi)^4}\left(\frac{1}{[\ell^2-\Delta]^2}-\frac{1}{[\ell^2-\Delta_\Lambda]^2}\right) &= \int \frac{d^4\ell}{(2\pi)^4}\frac{d}{d\Delta}\frac{d}{d\Delta_\Lambda}\left(\frac{1}{[\ell^2-\Delta]^2}-\frac{1}{[\ell^2-\Delta_\Lambda]^2}\right),\\
&= \int \frac{d^4\ell}{(2\pi)^4}\left(\frac{d}{d\Delta}\frac{1}{[\ell^2-\Delta]^2}-\frac{d}{d\Delta_\Lambda}\frac{1}{[\ell^2-\Delta_\Lambda]^2}\right),\\
&= 2\int \frac{d^4\ell}{(2\pi)^4}\left(\frac{1}{[\ell^2-\Delta]^2}-\frac{1}{[\ell^2-\Delta_\Lambda]^2}\right),\\
&= 2\frac{-i}{(4\pi)^2}\frac{1}{2}\left(\frac{1}{\Delta}-\frac{1}{\Delta_\Lambda}\right),\\
&= \frac{i}{(4\pi)^2}\left(\frac{1}{\Delta_\Lambda}-\frac{1}{\Delta}\right),\\
&= \frac{i}{(4\pi)^2}\frac{d}{d\Delta}\frac{d}{d\Delta_\Lambda}\log\left(\frac{\Delta_\Lambda}{\Delta}\right).
\end{aligned}$$

Because the differentiation clearly commutes with the constant factor, we have that

$$\therefore \int \frac{d^4\ell}{(2\pi)^4}\left(\frac{1}{[\ell^2-\Delta]^2}-\frac{1}{[\ell^2-\Delta_\Lambda]^2}\right)=\frac{i}{(4\pi)^2}\log\left(\frac{\Delta_\Lambda}{\Delta}\right). \tag{1.1}$$

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2. We are to find the roots of the simple quadratic,

$$(1-x)m_0^2+x\mu^2-x(1-x)p^2=x^2p^2-x(p^2+m_0^2-\mu^2)+m_0^2=0.$$

Invoking the quadratic formula, we see immediately that the roots are given by

$$\begin{aligned}
x &= \frac{p^2+m_0^2-\mu^2\pm\sqrt{(p^2+m_0^2-\mu^2)^2-4p^2m_0^2}}{2p^2},\\
&= \frac{1}{2}+\frac{m_0^2}{2p^2}-\frac{\mu^2}{2p^2}\pm\frac{1}{2p^2}\sqrt{p^4-2p^2(m_0^2+\mu^2)+(m_0^2-\mu^2)^2},\\
\therefore x &= \frac{1}{2}+\frac{m_0^2}{2p^2}-\frac{\mu^2}{2p^2}\pm\frac{1}{2p^2}\sqrt{[p^2-(m_0+\mu)^2][p^2-(m_0-\mu)^2]}.
\end{aligned} \tag{2.1}$$

3. We are to verify that when $p^2>(m_0^2+\mu^2)$ there is at least one real root of the equation where $x\in(0,1)$. First, we will show that the solutions are real. By checking the discriminant, we see that

$$[p^2-(m_0+\mu)^2][p^2-(m_0-\mu)^2]>[p^2-m_0^2-\mu^2][p^2-m_0^2-\mu^2+2m_0\mu]>1[1+2m_0\mu]>0.$$

Therefore the quadratic has only real roots. Now, let us show that the sum of the two solutions is positive. Noting that $\mu^2>0$, we have

$$x_1+x_2=1-\frac{m_0^2-\mu^2}{p^2}>1-\frac{m_0^2+\mu^2}{p^2}>0.$$

Therefore at least one of the two solutions must be positive. Lastly, we can show that the product of the two solutions is positive. This will guarantee that both solutions must be positive. By direct computation, we have

$$\begin{aligned} x_1x_2 &= \frac{1}{4p^4}\left(\left(p^2+m_0^2-\mu^2\right)^2-\left(p^2-(m_0+\mu)^2\right)\left(p^2-(m_0-\mu)^2\right)\right),\\ &= \frac{1}{4p^4}\left(\left(p^2+m_0^2-\mu^2\right)^2-\left(p^2-m_0^2-\mu^2-2m_0\mu\right)\left(p^2-m_0^2-\mu^2+2m_0\mu\right)\right),\\ &> \frac{1}{4p^4}\left(\left(p^2+m_0^2-\mu^2\right)^2-\left(p^2+m_0^2-\mu^2-2m_0\mu\right)\left(p^2+m_0^2-\mu^2+2m_0\mu\right)\right),\\ &= \frac{1}{4p^4}\left(\left(p^2+m_0^2-\mu^2\right)^2-\left(p^2+m_0^2-\mu^2\right)^2+4m_0\mu\right),\\ &= \frac{m_0\mu}{p^4}>0. \end{aligned}$$

Therefore, there are two real solutions to the equation. To show that a solution is confined to the interval $(0,1)$ we note that in the physically reasonable case where $\mu\to 0$, the x_2 solution becomes

$$\begin{aligned} x_2 &= \frac{1}{2p^2}\left(p^2+m_0^2-\sqrt{[p^2-m_0^2][p^2-m_0^2]}\right),\\ &= \frac{1}{2p^2}\left(p^2+m_0^2-\mu^2-p^2-m_0^2\right),\\ &= \frac{m_0^2}{p^2}<1. \end{aligned}$$

Therefore $x\in(0,1)$ is a real root of the quadratic equation of interest.

4. We are to show that $\delta F_1(0)+\delta Z_2=0$. To do this, we must first compute $\delta F_1(0)$. Let us recall the content of Peskin equation (6.47) while taking $q\to 0$,

$$\overline{u}(p')\delta\Gamma^\mu u(p) = 4ie^2\int_0^1 dxdydz\delta^{(3)}(x+y+z-1)\int\frac{d^4\ell}{(2\pi)^4}\frac{\overline{u}(p')\left[\gamma^\mu\cdot\left(-\frac{1}{2}\ell^2+(1-4z+z^2)m^2\right)\right]u(p)}{[\ell^2-\Delta]^3}.$$

We see that this term is just proportional to the $\delta F_1(0)$ term in our expression for $\delta\Gamma^\mu$. To actually compute this integral, we will require Pauli-Villars regularization of the term proportional to ℓ^2. Also, we will use the fact that $\lim_{\Lambda\to\infty}\Delta_\Lambda = z\Lambda^2$. Now, invoking the results of homework 2, we have that

$$\begin{aligned} \delta F_1(0) &= 4ie^2\int_0^1 dxdydz\delta^{(3)}(x+y+z-1)\int\frac{d^4\ell}{(2\pi)^4}\left[\left(-\frac{1}{2}\right)\left(\frac{\ell^2}{[\ell^2-\Delta]^3}-\frac{\ell^2}{[\ell^2-\Delta_\Lambda]^3}\right)+\frac{(1-4z+z^2)m^2}{[\ell^2-\Delta]^3}\right],\\ &= 4ie^2\int_0^1 dxdydz\delta^{(3)}(x+y+z-1)\left[\frac{-i}{2(4\pi)^2}\log\left(\frac{\Delta_\Lambda}{\Delta}\right)-\frac{i}{2(4\pi)^2}\frac{(1-4z+z^2)m^2}{\Delta}\right],\\ &= \frac{\alpha}{2\pi}\int_0^1 dxdydz\delta^{(3)}(x+y+z-1)\left[\log\left(\frac{\Delta_\Lambda}{\Delta}\right)+\frac{(1-4z+z^2)m^2}{\Delta}\right],\\ &= \frac{\alpha}{2\pi}\int_0^1 dxdydz\delta^{(3)}(x+y+z-1)\left[\log\left(\frac{z\Lambda^2}{(1-z)^2m^2+z\mu^2}\right)+\frac{(1-4z+z^2)m^2}{(1-z)^2m^2+z\mu^2}\right],\\ &= \frac{\alpha}{2\pi}\int_0^1 dz(1-z)\left[\log\left(\frac{z\Lambda^2}{(1-z)^2m^2+z\mu^2}\right)+\frac{(1-4z+z^2)m^2}{(1-z)^2m^2+z\mu^2}\right]. \end{aligned}$$

Quoting Peskin equation (7.31),

$$\delta Z_2 = \frac{\alpha}{2\pi}\int_0^1 dz\left[-z\log\left(\frac{z\Lambda^2}{(1-z)^2+z\mu^2}\right)+\frac{2z(2-z)(1-z)m^2}{(1-z)^2m^2+z\mu^2}\right].$$

Therefore,

$$\begin{aligned} \delta F_1(0)-\delta Z_2 &= \frac{\alpha}{2\pi}\int_0^1 dz\left[(1-2z)\log\left(\frac{z\Lambda^2}{(1-z)^2m^2+z\mu^2}\right)+\frac{(1-z)(1-4z+z^2)m^2+2z(2-z)(1-z)m^2}{(1-z)^2m^2+z\mu^2}\right],\\ &= \frac{\alpha}{2\pi}\int_0^1 dz\left[(1-2z)\log\left(\frac{z\Lambda^2}{(1-z)^2m^2+z\mu^2}\right)+\frac{m^2(z^3-z^2-z+1)}{(1-z)^2m^2+z\mu^2}\right]. \end{aligned}$$

To evaluate this integral, we will integrate the first part using integration by parts. Recall that, in general, $\left(\log \frac{f}{g}\right)' = \frac{f'}{f} - \frac{g'}{g} = \frac{f'g-g'f}{fg}$. Therefore, we may compute,

u		dv
$\log\left(\frac{z\Lambda^2}{(1-z)^2m^2+z\mu^2}\right)$	$\searrow^{+}$	$(1-2z)$
$\frac{m^2(1-z^2)}{z((1-z)^2m^2+z\mu^2)}$	$\overset{-}{\longleftarrow}$	$z-z^2$

$$\int_0^1 dz(1-2z)\log\left(\frac{z\Lambda^2}{(1-z)^2m^2+z\mu^2}\right),$$

$$= (z-z^2)\log\left(\frac{z\Lambda^2}{(1-z)^2m^2+z\mu^2}\right)\Bigg|_0^1 - \int_0^1 dz \frac{m^2(1-z^2)(z-z^2)}{z\left((1-z)^2m^2+z\mu^2\right)},$$

$$= 0 - \int_0^1 dz \frac{m^2(1-z^2)(z-z^2)}{z\left((1-z)^2m^2+z\mu^2\right)},$$

$$\therefore \int_0^1 dz(1-2z)\log\left(\frac{z\Lambda^2}{(1-z)^2m^2+z\mu^2}\right) = -\int_0^1 dz \frac{m^2(z^4-z^3-z^2+z))}{z\left((1-z)^2m^2+z\mu^2\right)}.$$

Therefore, we readily see that

$$\begin{aligned}
\delta F_1(0) - \delta Z_2 &= \frac{\alpha}{2\pi}\int_0^1 dz\left[(1-2z)\log\left(\frac{z\Lambda^2}{(1-z)^2m^2+z\mu^2}\right) + \frac{m^2(z^3-z^2-z+1)}{(1-z)^2m^2+z\mu^2}\right],\\
&= \frac{\alpha}{2\pi}\int_0^1 dz\left[\frac{m^2(z^4-z^3-z^2+z)}{z\left((1-z)^2m^2+z\mu^2\right)} + \frac{m^2(z^3-z^2-z+1)}{(1-z)^2m^2+z\mu^2}\right],\\
&= \frac{\alpha}{2\pi}\int_0^1 dz\left[\frac{m^2(z^4-z^3-z^2+z)}{z\left((1-z)^2m^2+z\mu^2\right)} + \frac{z}{z}\frac{m^2(z^3-z^2-z+1)}{(1-z)^2m^2+z\mu^2}\right],\\
&= \frac{\alpha}{2\pi}\int_0^1 dz\left[\frac{m^2(z^4-z^3-z^2+z)}{z\left((1-z)^2m^2+z\mu^2\right)} + \frac{m^2(z^4-z^3-z^2+z)}{z\left((1-z)^2m^2+z\mu^2\right)}\right],\\
&= 0.
\end{aligned}$$

$$\therefore \delta F_1(0) - \delta Z_2 = 0.$$

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Physics 523, Quantum Field Theory II

Homework 6

Due Wednesday, 18th February 2004

Jacob Lewis Bourjaily

Dimensional Regularization

a) We are to evaluate the expression

$$\int \frac{d^d\ell_E}{(2\pi)^d}\frac{1}{(\ell_E^2+\Delta)^n}, \quad \text{for } n \geq 2.$$

In homework 2, problem 3 we showed that the d-dimensional volume element $\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$. Using this, we see that

$$\begin{aligned}\int \frac{d^d\ell_E}{(2\pi)^d}\frac{1}{(\ell_E^2+\Delta)^n} &= \int \frac{d\Omega_d}{(2\pi)^d}\int_0^\infty d\ell_E \frac{\ell_E^{d-1}}{(\ell_E^2+\Delta)^n},\\ &= \frac{2\pi^{d/2}}{(2\pi)^d\Gamma(d/2)}\int_0^\infty d\ell_E\frac{\ell_E^{d-1}}{(\ell_E^2+\Delta)^n},\\ &= \frac{2}{(4\pi)^{d/2}}\frac{1}{\Gamma(d/2)}\int_0^\infty d\ell_E\frac{\ell_E^{d-1}}{(\ell_E^2+\Delta)^n},\\ &= \frac{1}{(4\pi)^{d/2}}\frac{1}{\Gamma(d/2)}\int_0^\infty d(\ell_E^2)\frac{(\ell_E^2)^{d/2-1}}{(\ell_E^2+\Delta)^n}.\end{aligned}$$

We will defined the integration variable

$$\eta \equiv \frac{\Delta}{(\ell_E^2+\Delta)} \quad \text{such that} \quad d\eta = -\frac{\Delta}{(\ell_E^2+\Delta)^2}d(\ell_E^2) \quad \text{and} \quad \ell_E^2 = \Delta\eta^{-1}(1-\eta).$$

Note that under the η substitution, the limits of integration will change from $(0,\infty)\mapsto(1,0)\sim -(0,1)$. Also note the use of the definition of the Euler Beta function below. Making this substitution in the required integral, we have

$$\begin{aligned}\int \frac{d^d\ell_E}{(2\pi)^d}\frac{1}{(\ell_E^2+\Delta)^n} &= \frac{1}{(4\pi)^{d/2}}\frac{1}{\Gamma(d/2)}\int_0^\infty d(\ell_E^2)\frac{(\ell_E^2)^{d/2-1}}{(\ell_E^2+\Delta)^n},\\ &= \frac{1}{(4\pi)^{d/2}}\frac{1}{\Gamma(d/2)}\frac{1}{\Delta}\int_0^1 d\eta\frac{\Delta^{d/2-1}\eta^{1-d/2}(1-\eta)^{d/2-1}}{(\ell_E^2+\Delta)^{n-2}},\\ &= \frac{1}{(4\pi)^{d/2}}\frac{1}{\Gamma(d/2)}\left(\frac{1}{\Delta}\right)^{2-d/2}\int_0^1 d\eta\left(\frac{\Delta}{\eta}\right)^{2-n}\eta^{1-d/2}(1-\eta)^{d/2-1},\\ &= \frac{1}{(4\pi)^{d/2}}\frac{1}{\Gamma(d/2)}\left(\frac{1}{\Delta}\right)^{2-d/2+n-2}\int_0^1 d\eta\eta^{n-2+1-d/2}(1-\eta)^{d/2-1},\\ &= \frac{1}{(4\pi)^{d/2}}\frac{1}{\Gamma(d/2)}\left(\frac{1}{\Delta}\right)^{n-d/2}\int_0^1 d\eta\eta^{n-d/2-1}(1-\eta)^{d/2-1},\\ &= \frac{1}{(4\pi)^{d/2}}\frac{1}{\Gamma(d/2)}\left(\frac{1}{\Delta}\right)^{n-d/2}\frac{\Gamma(n-d/2)\cdot\Gamma(d/2)}{\Gamma(n)},\end{aligned}$$

$$\boxed{\therefore \int \frac{d^d\ell_E}{(2\pi)^d}\frac{1}{(\ell_E^2+\Delta)^n} = \frac{1}{(4\pi)^{d/2}}\frac{\Gamma(n-d/2)}{\Gamma(n)}\left(\frac{1}{\Delta}\right)^{n-d/2}.} \tag{a.1}$$

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b) Let us now evaluate the expression

$$\int \frac{d^d\ell_E}{(2\pi)^d}\frac{\ell_E^2}{(\ell_E^2+\Delta)^n}, \quad \text{for } n \geq 2.$$

The evaluation of this integral will proceed identically to that in part (a) above. We will introduce the same integration variable $\eta \equiv \frac{\Delta}{(\ell_E^2+\Delta)}$ and follow the same procedure. We see that

$$\begin{aligned}
\int \frac{d^d\ell_E}{(2\pi)^d}\frac{\ell_E^2}{(\ell_E^2+\Delta)^n} &= \int \frac{d\Omega_d}{(2\pi)^d}\int_0^\infty d\ell_E \frac{\ell_E^{d+1}}{(\ell_E^2+\Delta)^n},\\
&= \frac{2\pi^{d/2}}{(2\pi)^d\Gamma(d/2)}\int_0^\infty d\ell_E \frac{\ell_E^{d+1}}{(\ell_E^2+\Delta)^n},\\
&= \frac{1}{(4\pi)^{d/2}}\frac{1}{\Gamma(d/2)}\int_0^\infty d(\ell_E^2)\frac{(\ell_E^2)^{d/2}}{(\ell_E^2+\Delta)^n},\\
&= \frac{1}{(4\pi)^{d/2}}\frac{1}{\Gamma(d/2)}\frac{1}{\Delta}\int_0^1 d\eta \frac{\Delta^{d/2}\eta^{-d/2}(1-\eta)^{d/2}}{(\ell_E^2+\Delta)^{n-2}},\\
&= \frac{1}{(4\pi)^{d/2}}\frac{1}{\Gamma(d/2)}\left(\frac{1}{\Delta}\right)^{1-d/2}\int_0^1 d\eta\left(\frac{\Delta}{\eta}\right)^{2-n}\eta^{-d/2}(1-\eta)^{d/2},\\
&= \frac{1}{(4\pi)^{d/2}}\frac{1}{\Gamma(d/2)}\left(\frac{1}{\Delta}\right)^{1-d/2+n-2}\int_0^1 d\eta\eta^{n-1-d/2-1}(1-\eta)^{d/2+1-1},\\
&= \frac{1}{(4\pi)^{d/2}}\frac{1}{\Gamma(d/2)}\left(\frac{1}{\Delta}\right)^{n-d/2-1}\frac{\Gamma(n-1-d/2)\cdot\Gamma(d/2+1)}{\Gamma(n)}.
\end{aligned}$$

Recall the elementary property of the Γ function that $\alpha\Gamma(\alpha)=\Gamma(\alpha+1)$. Therefore we see that $\Gamma(d/2+1)=\frac{d}{2}\Gamma(d/2)$. Using this result, we see immediately that

$$\boxed{\therefore \int \frac{d^d\ell_E}{(2\pi)^d}\frac{\ell_E^2}{(\ell_E^2+\Delta)^n} = \frac{1}{(4\pi)^{d/2}}\frac{d}{2}\frac{\Gamma\left(n-\frac{d}{2}-1\right)}{\Gamma(n)}\left(\frac{1}{\Delta}\right)^{n-d/2-1}.} \tag{b.1}$$

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c i) Let us show the following identity,

$$\gamma^\mu\gamma^\nu\gamma_\mu = -(2-\epsilon)\gamma^\nu.$$

Simply applying the anticommutation relation of the γ matrices, we see that[1]

$$\gamma^\mu\gamma^\nu\gamma_\mu = g_{\mu\rho}\gamma^\mu\gamma^\nu\gamma^\rho = 2g_{\mu\rho}g^{\nu\rho}\gamma^\mu - g_{\mu\rho}\gamma^\mu\gamma^\rho\gamma^\nu = 2\delta^\nu_\mu\gamma^\mu - d\gamma^\nu = (2-d)\gamma^\nu,$$

$$\boxed{\therefore \gamma^\mu\gamma^\nu\gamma_\mu = -(2-\epsilon)\gamma^\nu.} \tag{c.1}$$

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c ii) Let us show the following identity,

$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 4g^{\nu\rho} - \epsilon\gamma^\nu\gamma^\rho.$$

Simply applying the anticommutation relation of the γ matrices, we see that

$$\begin{aligned}
\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu &= g_{\mu\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = 2g_{\mu\sigma}g^{\rho\sigma}\gamma^\mu\gamma^\nu - 2g_{\mu\sigma}g^{\mu\rho} + g_{\mu\sigma}\gamma^\mu\gamma^\sigma\gamma^\nu\gamma^\rho,\\
&= 2\delta^\rho_\mu\gamma^\mu\gamma^\nu - 2\delta^\sigma_\mu\gamma^\mu\gamma^\rho + d\gamma^\nu\gamma^\rho = 2\gamma^\rho\gamma^\nu + 2\gamma^\nu\gamma^\rho - 4\gamma^\nu\gamma^\rho + d\gamma^\nu\gamma^\rho,
\end{aligned}$$

$$\boxed{\therefore \gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 4g^{\nu\rho} - \epsilon\gamma^\nu\gamma^\rho.} \tag{c.2}$$

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[1] We will repeatedly use that $g_{\mu\rho}\gamma^\mu\gamma^\rho\mathscr{X} = g_{\mu\rho}\gamma^\rho\gamma^\mu\mathscr{X}$ by symmetry of the inner product together with $g_{\mu\rho}\gamma^\mu\gamma^\rho\mathscr{X} = 2g_{\mu\rho}g^{\mu\rho}\mathscr{X} - g_{\mu\rho}\gamma^\rho\gamma^\mu\mathscr{X}$ from the anticommutation relations, imply that $g_{\mu\rho}\gamma^\mu\gamma^\rho\mathscr{X} = g_{\mu\rho}g^{\mu\rho}\mathscr{X} = d\mathscr{X}$ for any product of γ matrices $\mathscr{X}$.

c iii) Let us show the following identity,

$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu = -2\gamma^\sigma\gamma^\rho\gamma^\nu + \epsilon\gamma^\nu\gamma^\rho\gamma^\sigma.$$

Simply applying the anticommutation relation of the γ matrices, we see that

$$\begin{aligned}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu &= g_{\mu\tau}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\tau = g_{\mu\tau}\left(2g^{\sigma\tau}\gamma^\mu\gamma^\nu\gamma^\rho - 2g^{\rho\tau}\gamma^\mu\gamma^\nu\gamma^\sigma + 2g^{\nu\tau}\gamma^\mu\gamma^\rho\gamma^\sigma - g^{\mu\tau}\gamma^\nu\gamma^\rho\gamma^\sigma\right),\\ &= 2\delta^\sigma_\mu\gamma^\mu\gamma^\nu\gamma^\rho - 2\delta^\rho_\mu\gamma^\mu\gamma^\nu\gamma^\sigma + 2\delta^\nu_\mu\gamma^\mu\gamma^\rho\gamma^\sigma - d\gamma^\nu\gamma^\rho\gamma^\sigma = 2\gamma^\sigma\gamma^\nu\gamma^\rho - 4g^{\nu\rho}\gamma^\sigma + (4-d)\gamma^\nu\gamma^\rho\gamma^\sigma,\\ &= 4g^{\mu\rho}\gamma^\sigma - 2\gamma^\sigma\gamma^\rho\gamma^\nu - 4g^{\nu\rho}\gamma^\sigma + \epsilon\gamma^\nu\gamma^\rho\gamma^\sigma,\end{aligned}$$

$$\boxed{\therefore \gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu = -2\gamma^\sigma\gamma^\rho\gamma^\nu + \epsilon\gamma^\nu\gamma^\rho\gamma^\sigma.} \tag{c.3}$$

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The Ward Identity

a i) Let us compute the integral

$$\int\frac{d^4\ell}{(2\pi)^4}\frac{1}{(\ell^2-\Delta)^2},$$

by restricting the integration region to the Euclidean sphere with $\ell_E < \Lambda$. To accomplish this calculation, we will recall several important results from earlier homework problems. Namely, we will use the standard 4-dimensional volume element and change to Euclidean coordinates ℓ_E. Notice the u substitution below.

$$\begin{aligned}\int\frac{d^4\ell}{(2\pi)^4}\frac{1}{(\ell^2-\Delta)^2} &= \int\frac{d\Omega_4}{(2\pi)^4}\int_0^\infty d\ell\frac{\ell^3}{(\ell^2-\Delta)^2},\\ &= \frac{2i}{(4\pi)^2}\int_0^\infty d\ell_E\frac{\ell_E^3}{(\ell_E^2+\Delta)^2},\\ &\to \frac{2i}{(4\pi)^2}\lim_{\Lambda\to\infty}\int_0^\Lambda d\ell_E\frac{\ell_E^3}{(\ell_E^2+\Delta)^2},\\ &= \frac{i}{(4\pi)^2}\lim_{\Lambda\to\infty}\int_\Delta^{\Lambda^2+\Delta}du\frac{u-\Delta}{u^2},\\ &= \frac{i}{(4\pi)^2}\lim_{\Lambda\to\infty}\left[\log(u)\Big|_\Delta^{\Lambda^2+\Delta} + \frac{\Delta}{u}\Big|_\Delta^{\Lambda^2+\Delta}\right],\\ &= \frac{i}{(4\pi)^2}\lim_{\Lambda\to\infty}\left[\log\left(\frac{\Lambda^2+\Delta}{\Delta}\right)-1\right],\\ &= \frac{i}{(4\pi)^2}\left[\log\left(\frac{\Lambda^2}{\Delta}\right)-1+\mathscr{O}\left(\frac{\Delta}{\Lambda^2}\right)\right].\end{aligned}$$

ii) We are to compute the function Z_1 from the $\delta\Gamma(q=0)$ calculation. Recall that in homework 5 question 4, we computed $\delta F_1(q=0)$ using a different regularization. Because $\delta Z_1 = -\delta F_1(q=0)$ much of our 'hard labor' has already been completed. Let us begin our calculation.

$$\begin{aligned}\delta Z_1 &= -4ie^2\int_0^1 dz(1-z)\int\frac{d^4\ell}{(2\pi)^4}\left[-\frac{1}{2}\frac{\ell^2}{(\ell^2-\Delta)^3}+\frac{m^2(1-4z+z^2)}{(\ell^2-\Delta)^3}\right],\\ &= -4ie^2\int_0^1 dz(1-z)\int\frac{d^4\ell}{(2\pi)^4}\left[-\frac{1}{2}\left(\frac{1}{(\ell^2-\Delta)^3}+\frac{\Delta}{(\ell^2-\Delta)^3}\right)+\frac{m^2(1-4z+z^2)}{(\ell^2-\Delta)^3}\right],\\ &= -4ie^2\int_0^1 dz(1-z)\left[-\frac{1}{2}\frac{i}{(4\pi)^2}\left(\log\left(\frac{\Lambda^2}{\Delta}\right)-1+\mathscr{O}\left(\frac{\Delta}{\Lambda^2}\right)\right)+\frac{1}{4}\frac{i}{(4\pi)^2}-\frac{1}{2}\frac{i}{(2\pi)^2}\frac{m^2(1-4z+z^2)}{\Delta}\right],\\ &= -\frac{\alpha}{4\pi}\int_0^1 dz(1-z)\left[\log\left(\frac{\Lambda^2}{\Delta}\right)-1-\frac{1}{2}+\frac{m^2(1-4z+z^2)}{\Delta}\right],\end{aligned}$$

$$\therefore \delta Z_1 = -\frac{\alpha}{4\pi}\int_0^1\log\left(\frac{\Lambda^2}{\Delta}\right)-\frac{3}{2}+\frac{m^2(1-4z+z^2)}{\Delta}.$$

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iii) Let us now compute the value of the electron self-energy function Z_d. First, we must recall the definition of Z_2. It is the function

$$\delta Z_2 = \left.\frac{d\Sigma_2}{d\not p}\right|_{\not p=m}$$

where

$$\Sigma_2 = -ie^2\int_0^1 dz \int \frac{d^4\ell}{(2\pi)^4}\frac{-2z\not p+4m}{(\ell^2-\Delta)^2} = \frac{\alpha}{2\pi}\int_0^1 dz(2m-z\not p)\left[\log\left(\frac{\Lambda^2}{\Delta}\right)-1\right].$$

Using the chain rule for differentiation, we see that

$$\therefore \delta Z_2 = \frac{\alpha}{2\pi}\int_0^1 dz\left[-z\left(\log\left(\frac{\Lambda^2}{\Delta}\right)-1\right)+\frac{2m^2z(2-z)(1-z)}{\Delta}\right].$$

iv) We will now compute the difference $Z_2 - Z_1 = \delta Z_2 - \delta Z_1$ for this regularization scheme. We will call upon Peskin and Schroeder for algebraic simplification within the integrand. The cancellation of the log-type term with the $1/\Delta$ term was shown in homework 5. We have

$$\begin{aligned}\delta Z_2 - \delta Z_1 &= \frac{\alpha}{2\pi}\int_0^1 dz\left[(1-2z)\log\left(\frac{\Lambda^2}{\Delta}\right)+z-\frac{3}{2}(1-z)+\frac{2m^2z(2-z)(1-z)}{\Delta}-\frac{m^2(1-4z+z^2)}{\Delta}\right],\\ &= \frac{\alpha}{2\pi}\int_0^1 dz\left[z-\frac{3}{2}(1-z)\right] = \frac{\alpha}{2\pi}\left(-\frac{1}{4}\right),\end{aligned}$$

$$\therefore \delta Z_2 - \delta Z_1 = -\frac{\alpha}{8\pi}.$$

b i) Let us repeat our above calculation using dimensional regularization. We can begin our work by generalizing the Dirac algebra used to calculate δZ_1. Notice that this calculation will require our $d = 4-\epsilon$ dimensional generalization of the Dirac algebra to simplify the numerator in

$$\delta\Gamma^\mu(q^2=0) = 2ie^2\int_0^1 dz(1-z)\int\frac{d^d\ell}{(2\pi)^d}\frac{\gamma^\nu(\not\ell+z\not p)\gamma^\mu(\not\ell+z\not p)\gamma_\nu}{(\ell^2-\Delta)^3}.$$

Although we have already simplified our work by leaving off terms proportional to q, we may reduce our labor even more. The regularization of this integral in d-dimensions is presented to make sense of the divergence of the integral. Computing the integral in $d = 4-\epsilon$ dimensions, we avoid the divergence of the integral due to the term proportional to ℓ^2 in the numerator. However, we should notice that no other terms in the integral will have a power of $\ell \leq 4$ in the denominator and therefore will not diverge.

Therefore, only the ℓ^2-term will need to be regulated and the other parts of this integral can be computed as usual.[2]

Let us then compute the regulated coefficient of the ℓ^2 term in the the numerator. To do this, we will use our algebraic results from problem (1.c.iii) above. We also remind the reader that in d-dimensions the integral is symmetric under $\ell^\mu\ell^\nu \to \frac{1}{d}\gamma^{\mu\nu}\ell^2$. Therefore we see that our regulated term is simply

$$\begin{aligned}\gamma^\nu\not\ell\gamma^\mu\not\ell\gamma_\nu &= \ell_\rho\ell_\sigma\gamma^\nu\gamma^\rho\gamma^\mu\gamma^\sigma\gamma_\nu,\\ &= \ell_\rho\ell_\sigma\left(-2\gamma^\sigma\gamma^\mu\gamma^\rho+\epsilon\gamma^\rho\gamma^\mu\gamma^\sigma\right),\\ &= -4\not\ell\ell^\mu+2\not\ell^2\gamma^\mu+2\epsilon\not\ell\ell^\mu-\epsilon\not\ell^2\gamma^\mu,\\ &= -\frac{4}{d}\ell^2\gamma^\mu+2\ell^2\gamma^\mu+\frac{2\epsilon}{d}\ell^2\gamma^\mu-\epsilon\ell^2\gamma^\mu,\\ &= \gamma^\mu\ell^2\left(\frac{-4+2\epsilon}{d}+2-\epsilon\right),\\ &= \gamma^\mu\ell^2\frac{(\epsilon-2)^2}{d}.\end{aligned}$$

---

[2] It makes little sense to regulate a convergent integral. More rigorously, one could carry ϵ dependence on all terms and then 'observe' that for all but the term proportional to ℓ^2 in the numerator, $\epsilon \to 0$ will not affect the integral. Therefore we may view the introduction of ϵ into those terms as a waste of time.

Now that we have fully established the need only to regularize this piece of the integral, let us calculate the regularized form of δZ_1. During the computation below, we have referred to the canonical results for expansions of $\Delta, \Gamma, \frac{1}{(4\pi)}$ in terms of ϵ. Many of these relations were derived in homework sets 2 and 5. Let us proceed directly.

$$\begin{aligned}
\delta Z_1 &= -2ie^2 \int_0^1 dz(1-z) \int \frac{d^d\ell}{(2\pi)^d} \left[\frac{(\epsilon-2)^2}{d} \frac{\ell^2}{(\ell^2-\Delta)^3} + \frac{m^2(1-4z+z^2)}{(\ell^2-\Delta)^3} \right], \\
&= -2ie^2 \int_0^1 dz(1-z) \left[\frac{(\epsilon-2)^2}{d} \frac{d}{4} \frac{i}{(4\pi)^{d/2}} \frac{\Gamma\left(2-\frac{d}{2}\right)}{\Delta^{2-d/2}} - \frac{i}{2} \frac{1}{(4\pi)^2} \frac{m^2(1-4z+z^2)}{\Delta} \right], \\
&= \frac{\alpha}{2\pi} \int_0^1 dz(1-z) \left[\frac{(\epsilon-2)^2}{4} \left(\frac{2}{\epsilon} - \log\Delta - \gamma_E + \log(4\pi) \right) - \frac{1}{2} \frac{m^2(1-4z+z^2)}{\Delta} \right] \\
\therefore \delta Z_1 &= \frac{\alpha}{2\pi} \int_0^1 dz(1-z) \left[-\left(\frac{2}{\epsilon} - 2 - \log\Delta - \gamma_E + \log(4\pi) \right) - \frac{1}{2} \frac{m^2(1-4z+z^2)}{\Delta} \right].
\end{aligned}$$

ii) Let us now regularize the term Z_2. This computation will be very similar to that above. We will first need to rework some minor Dirac algebra. Unlike last time, however, the entire integral will diverge and so we will need to keep ϵ terms consistently in our equations. Recall that Z_2 is related to a derivative of the integral

$$\Sigma_2(p) = -ie^2 \int_0^1 dz \int \frac{d^d\ell}{(2\pi)^d} \frac{\gamma^\mu \left(\not\ell + z\not p + m\right)\gamma_\mu}{(\ell^2-\Delta)^2}.$$

Recalling that terms proportional to ℓ in the integral will integrate to zero because of Lorentz covariance, we may drop the ℓ term. Furthermore, using only the relatively trivial Dirac algebra identities derived above, we see that

$$\gamma^\mu \left(\not\ell + z\not p + m\right)\gamma_\mu \to -z(2-\epsilon)\not p + dm.$$

Therefore we may compute this integral directly.

$$\begin{aligned}
\Sigma_2(p) &= -ie^2 \int_0^1 dz \int \frac{d^d\ell}{(2\pi)^d} \left(-(2-\epsilon)z\not p + (4-\epsilon)m\right) \frac{1}{(\ell^2-\Delta)^2}, \\
&= -ie^2 \int_0^1 dz \left[\left(-(2-\epsilon)z\not p + (4-\epsilon)m\right) \frac{i}{(4\pi)^{d/2}} \frac{\Gamma\left(2-\frac{d}{2}\right)}{\Delta^{2-d/2}} \right], \\
&= \frac{\alpha}{2\pi} \int_0^1 dz \left[\frac{1}{2}\left((4-\epsilon)m - (2-\epsilon)z\not p\right) \left(\frac{2}{\epsilon} - \log\Delta - \gamma_E + \log(4\pi) \right) \right].
\end{aligned}$$

Therefore we see by simple chain-rule differentiation that

$$\begin{aligned}
\delta Z_2 = \left.\frac{d\Sigma_2}{d\not p}\right|_{\not p=m} &= \frac{\alpha}{2\pi} \int_0^1 dz \frac{1}{2} \left[(\epsilon-2)z \left(\frac{2}{\epsilon} - \log\Delta - \gamma_E + \log(4\pi) \right) - \frac{m^2 2z(1-z)\left((\epsilon-2)z + (4-\epsilon)\right)}{\Delta} \right], \\
&= \frac{\alpha}{2\pi} \int_0^1 dz \left[z\left(1 - \frac{2}{\epsilon} + \log\Delta + \gamma_E - \log(4\pi) \right) - \frac{m^2 2z(1-z)(2-z)}{\Delta} \right],
\end{aligned}$$

iii) Unfortunately, I was unable to derive the explicit cancellation. It appears as if I may have introduced an incorrect minus sign somewhere. In the correct form, one should see the total integral vanish so that

$$\delta Z_2 - \delta Z_1 = 0.$$

Physics 523, Quantum Field Theory II

Homework 7

Due Wednesday, 3[rd] March 2004

Jacob Lewis Bourjaily

Superficial Divergences

Let us consider φ^3 scalar field theory in $d = 4$ dimension. The Lagrangian for this theory is

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{2}m^2\varphi^2 - \frac{1}{3!}g\varphi^3.$$

a) Let us determine the superficial divergence D for this theory in terms of the number of vertices V and the number of external lines N. From this we are to show that the theory is super-renormalizable.

In generality, the superficial divergence of a φ^n theory in d dimensions can be given by $D = dL - 2P$, where L is the number of loops and P is the number of propagators because each loop contributes a d-dimensional integration and each propagator contributes a power of 2 in the denominator. Furthermore, we see that $nV = N + 2P$ because each external line connects to one vertex and each propagator connects two and each vertex involves n lines. This implies that $P = \frac{1}{2}(nV - N)$.

Therefore, still in complete generality, the superficial divergence of a φ^n theory in d-dimensions may be written

$$D = dL - 2P = \frac{d}{2}nV - \frac{d}{2}N - dV + d - nV + N,$$
$$= d + \left(n\frac{d-2}{2} - d\right)V - \frac{d-2}{2}N.$$

Therefore, in a 4-dimensional φ^3-theory the superficial divergence is given by

$$\boxed{D = 4 - V - N.} \qquad (1.a.1)$$

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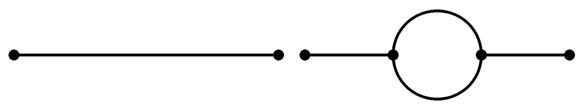
We see that because $D \propto -V$ the theory is *super-renormalizable.*

b) We are to show the superficially divergent diagrams for this theory that are associated with the exact two-point function.

Using equation (1.a) above, we see that the three superficially divergent diagrams in this φ^3-theory associated with the exact two-point function are:



c) Let us compute the mass dimension of the coupling constant g.

Because $\mathscr{L}$ must have dimension $(\text{mass})^4$ each term should have dimension $(\text{mass})^4$. Because of the $m^2\varphi^2$ term, this implies that the field φ has dimension $(\text{mass})^1$. Therefore the coupling g must have dimension $(\text{mass})^1$.

Renormalization and the Yukawa Coupling

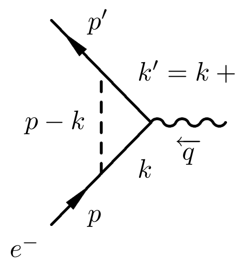
We are to consider the theory of elementary fermions that couple to both QED and a Yukawa field ϕ governed by the interaction Hamiltonian

$$H_{\text{int}} = \int d^3x \frac{\lambda}{\sqrt{2}}\phi\overline{\psi}\psi + \int d^3x e A_\mu \overline{\psi}\gamma^\mu\psi.$$

a) Let us verify that $\delta Z_1 = \delta Z_2$ to the one-loop order.

We computed in homework 4 the amplitude for the $\overline{\psi}\gamma\psi$ vertex with a virtual scalar ϕ,

$i\mathscr{M} =$



$$= \int \frac{d^4k}{(2\pi)^4}\overline{u}(p')\frac{-i\lambda}{\sqrt{2}}\frac{i}{((p-k)^2 - m_\phi^2 + i\epsilon)}\frac{i(\not{k}' + m)}{(k'^2 - m^2 + i\epsilon)}(-ie\gamma^\mu)\frac{i(\not{k} + m)}{(k^2 - m^2 + i\epsilon)}\frac{-i\lambda}{\sqrt{2}}u(p),$$

In the limit where $q \to 0$, we see that this implies

$$\overline{u}(p)\delta\Gamma^{\mu}u(p) = i\frac{\lambda^2}{2}\int\frac{d^dk}{(2\pi)^d}\frac{\overline{u}(p)\left[(\not{k}+m)\,\gamma^{\mu}\,(\not{k}+m)\right]u(p)}{((p-k)^2-m_{\phi}^2+i\epsilon)(k^2-m^2+i\epsilon)(k^2-m^2+i\epsilon)}.$$

Using Feynman parametrization to simplify the denominator, we will use the variables

$$\ell \equiv k - zp \qquad \text{and} \qquad \Delta \equiv (1-z)^2m^2 + zm_{\phi}^2.$$

The numerator of the integrand is then reduced to

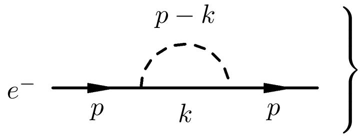
$$\begin{aligned}
\mathcal{N} &= \overline{u}(p)\left[(\not{\ell}+z\not{p}+m)\,\gamma^{\mu}\,(\not{\ell}+z\not{p}+m)\right]u(p),\\
&= \overline{u}(p)\left[\not{\ell}\gamma^{\mu}\not{\ell}+z^2\not{p}\gamma^{\mu}\not{p}+mz\not{p}\gamma^{\mu}+mz\gamma^{\mu}\not{p}+m^2\gamma^{\mu}\right]u(p),\\
&= \overline{u}(p)\left[\frac{1}{d}\ell^2(2\gamma^{\mu}-d\gamma^{\mu})+z^2m^2\gamma^{\mu}+m^2z\gamma^{\mu}+m^2z\gamma^{\mu}+m^2\gamma^{\mu}\right]u(p),\\
&= \overline{u}(p)\left[\gamma^{\mu}\left(\frac{2-d}{d}\ell^2+m^2(1+z)^2\right)\right]u(p).
\end{aligned}$$

Combining this with our work above, we see that this implies

$$\begin{aligned}
\delta Z_1 = -\delta F_1(q=0) &= -i\frac{\lambda^2}{2}\int_0^1 dz(1-z)2\int\frac{d^d\ell}{(2\pi)^d}\left[\frac{\left(\frac{2-d}{d}\right)\ell^2}{[\ell^2-\Delta+i\epsilon]^3}+\frac{m^2(1+z)^2}{[\ell^2-\Delta+i\epsilon]^3}\right],\\
&= -i\frac{\lambda^2}{2}\int_0^1 dz(1-z)\left[\frac{2-d}{d}\frac{d}{2}\frac{i}{(4\pi)^{d/2}}\frac{\Gamma\left(2-\frac{d}{2}\right)}{\Delta^{2-d/2}}-\frac{i}{(4\pi)^2}\frac{m^2(1+z)^2}{\Delta}\right],\\
&\simeq \frac{\lambda^2}{32\pi^2}\int_0^1 dz(1-z)\left[\frac{2-d}{2}\left(\frac{2}{\epsilon}-\log\Delta-\gamma_E+\log(4\pi)\right)-\frac{m^2(1+z)^2}{\Delta}\right],\\
&= \frac{\lambda^2}{32\pi^2}\int_0^1 dz(1-z)\left[\frac{\epsilon-2}{2}\left(\frac{2}{\epsilon}-\log\Delta-\gamma_E+\log(4\pi)\right)-\frac{m^2(1+z)^2}{\Delta}\right],
\end{aligned}$$

$$\boxed{\therefore\,\delta Z_1 = \frac{\lambda^2}{32\pi^2}\int_0^1 dz(1-z)\left[1-\left(\frac{2}{\epsilon}-\log\Delta-\gamma_E+\log(4\pi)\right)-\frac{m^2(1+z)^2}{\Delta}\right]} \qquad (2.a.1)$$

Let us now compute the one-loop contribution of ϕ to the electron two-point function,



$$\Longrightarrow \Sigma_{\phi_2} = \frac{\lambda^2}{2}\int\frac{d^4k}{(2\pi)^4}\frac{i(\not{k}+m)}{((p-k)^2-m_{\phi}^2+i\epsilon)(k^2-m^2+i\epsilon)}$$

We will define the following variables for Feynman parametrization of the denominator:

$$\ell \equiv k - zp, \qquad \text{and} \qquad \Delta \equiv -z(1-z)\not{p}^2 + zm_{\phi}^2 + (1-z)m^2.$$

We see therefore that

$$\begin{aligned}
\Sigma_{\phi_2} &= i\frac{\lambda^2}{2}\int_0^1 dz\int\frac{d^d\ell}{(2\pi)^d}\frac{z\not{p}+m}{[\ell^2-\Delta+i\epsilon]^2},\\
&= i\frac{\lambda^2}{2}\int_0^1 dz(z\not{p}+m)\frac{i}{(4\pi)^{d/2}}\frac{\Gamma\left(2-\frac{d}{2}\right)}{\Delta^{2-d/2}},\\
&\simeq -\frac{\lambda^2}{32\pi^2}\int_0^1 dz(z\not{p}+m)\left(\frac{2}{\epsilon}-\log\Delta-\gamma_E+\log(4\pi)\right).
\end{aligned}$$

Therefore,

$$\delta Z_2 = \left.\frac{\partial\Sigma_{\phi_2}}{\partial\not{p}}\right|_{\not{p}=m} = -\frac{\lambda^2}{32\pi^2}\int_0^1 dz\left[z\left(\frac{2}{\epsilon}-\log\Delta-\gamma_E+\log(4\pi)\right)+(zm+m)\frac{2mz(1-z)}{\Delta}\right],$$

$$\boxed{\therefore\,\delta Z_2 = -\frac{\lambda^2}{32\pi^2}\int_0^1 dz\left[z\left(\frac{2}{\epsilon}-\log\Delta-\gamma_E+\log(4\pi)\right)+\frac{2m^2z(1+z)(1-z)}{\Delta}\right].} \qquad (2.a.2)$$

Let us now compute the difference $\delta Z_2 - \delta Z_1$. We see that

$$\delta Z_2 - \delta Z_1 = \frac{\lambda^2}{32\pi^2}\int_0^1 dz \left[(1-2z)\log\left(\frac{1}{\Delta}\right) + (1-2z)\left(\frac{2}{\epsilon} - \gamma_E + \log(4\pi)\right) - (1-z) - \frac{m^2(1-z)(1+z)}{\Delta}\left(2z-(1+z)\right)\right],$$
$$= \frac{\lambda^2}{32\pi^2}\int_0^1 dz \left[(1-2z)\log\left(\frac{1}{\Delta}\right) - (1-z) + \frac{m^2(1-z)^2(1+z)}{\Delta}\right],$$
$$= \frac{\lambda^2}{32\pi^2}\int_0^1 dz \left[(1-z) - \frac{m^2(1-z)(1-z^2)}{\Delta} - (1-z) + \frac{m^2(1-z)^2(1+z)}{\Delta}\right],$$
$$= \frac{\lambda^2}{32\pi^2}\int_0^1 dz \left[-\frac{m^2(1-z)^2(1+z)}{\Delta} + \frac{m^2(1-z)^2(1+z)}{\Delta}\right],$$
$$\boxed{\therefore \delta Z_2 - \delta Z_1 = 0.} \tag{2.a.3}$$

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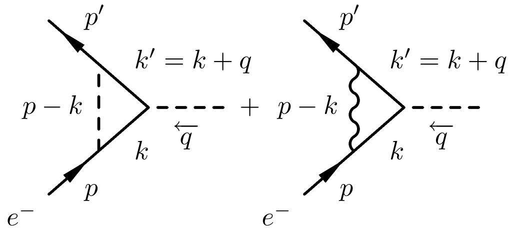
We can expect that $Z_1 = Z_2$ quite generally in this theory because our proof of the Ward-Takahashi identity relied, fundamentally, on the local $U(1)$ gauge invariance of the A_μ term in the Lagrangian which is not altered by the addition of the scalar ϕ.

b) Let us now consider the renormalization of the $\overline{\psi}\phi\psi$ vertex in this theory.
The two diagrams at the one-loop level that contribute to $\overline{u}(p')\delta\Gamma u(p)$ are



These diagrams yield

$$\overline{u}(p')\delta\Gamma u(p) = \int\frac{d^dk}{(2\pi)^d}\overline{u}(p')\left[\left(-i\frac{\lambda}{\sqrt{2}}\right)\frac{i}{((p-k)^2 - m_\phi^2 + i\epsilon)}\frac{i(\not{k}+\not{q}+m)}{((k+q)^2 - m^2 + i\epsilon)}\frac{i(\not{k}+m)}{(k^2-m^2+i\epsilon)}\left(-i\frac{\lambda}{\sqrt{2}}\right)\right.$$
$$\left. + (-ie\gamma^\mu)\frac{i(\not{k}+\not{q}+m)}{((k+q)^2-m^2)}\frac{-i}{((p-k)^2-\mu^2)}\frac{i(\not{k}+m)}{(k^2-m^2)}(-ie\gamma_\mu)\right]u(p).$$

Taking the limit where $q \to 0$ and introducing the variables

$$\ell \equiv k - zp, \qquad \Delta_1 \equiv (1-z)^2m^2 + zm_\phi^2, \quad \text{and} \quad \Delta_2 \equiv (1-z)^2m^2 + z\mu^2,$$

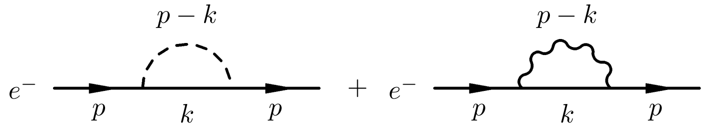
this becomes,

$$\overline{u}(p)\delta\Gamma u(p) = \int_0^1 dz(1-z)\int\frac{d^d\ell}{(2\pi)^d}\overline{u}(p)\left[i\lambda^2\frac{\ell^2 + (1+z)^2m^2}{(\ell^2-\Delta_1+i\epsilon)^3} - 2ie^2\frac{d\ell^2 + m^2\left(d(z^2+1)+2z(2-d)\right)}{(\ell^2-\Delta_2+i\epsilon)^3}\right]u(p).$$

Therefore,

$$\delta Z_1' = -\delta F_1' = \int_0^1 dz(1-z)\int\frac{d^d\ell}{(2\pi)^d}\left[-i\lambda^2\frac{\ell^2+(1+z)^2m^2}{(\ell^2-\Delta_1+i\epsilon)^3} + 2ie^2\frac{d\ell^2 + m^2\left(d(z^2+1)+2z(2-d)\right)}{(\ell^2-\Delta_2+i\epsilon)^3}\right],$$
$$= \int_0^1 dz(1-z)\int\frac{d^d\ell}{(2\pi)^d}\left[-i\lambda^2\frac{\ell^2}{(\ell^2-\Delta_1+i\epsilon)^3} + 2ie^2\frac{d\ell^2}{(\ell^2-\Delta_2+i\epsilon)^3}\right] + \text{ finite terms},$$
$$= \int_0^1 dz(1-z)\left[\frac{\lambda^2}{4}\frac{d}{(4\pi)^{d/2}}\frac{\Gamma\left(2-\frac{d}{2}\right)}{\Delta_1^{2-d/2}} - \frac{e^2}{2}\frac{d^2}{(4\pi)^{d/2}}\frac{\Gamma\left(2-\frac{d}{2}\right)}{\Delta_2^{2-d/2}}\right] + \text{ finite terms},$$
$$= \int_0^1 dz(1-z)\left[\frac{\lambda^2}{16\pi^2}\left(\frac{2}{\epsilon} - \log\Delta_1 - \gamma_E + \log(4\pi) - \frac{1}{2}\right) - \frac{2\alpha}{\pi}\left(\frac{2}{\epsilon} - \log\Delta_2 - \gamma_E + \log(4\pi) - 1\right)\right] + \text{ finite terms},$$
$$= \int_0^1 dz(1-z)\frac{2}{\epsilon}\left(\frac{\lambda^2}{16\pi^2} - \frac{2\alpha}{\pi}\right) + \text{ finite terms},$$
$$\boxed{\therefore \delta Z_1' = \frac{1}{\epsilon}\left(\frac{\lambda^2}{16\pi^2} - \frac{2\alpha}{\pi}\right) + \text{ finite terms.}} \tag{2.b.2}$$

Now let us compute $\delta Z_2'$. We see that this factor comes from the diagrams,



We see that we have already computed both of these contributions; the first diagram's contribution was computed above and the second diagram's contribution was computed in homework 6.

Therefore, we note that

$$\boxed{\delta Z_2' = \frac{1}{\epsilon}\left(-\frac{\lambda^2}{32\pi^2} - \frac{\alpha}{2\pi}\right) + \text{ finite terms.}} \tag{2.b.3}$$

Combining these results, we have that

$$\boxed{\therefore \delta Z_2' - \delta Z_1' = \frac{3}{\epsilon}\left(\frac{\alpha}{2\pi} - \frac{\lambda^2}{32\pi^2}\right) + \text{ finite terms} \neq 0.} \tag{2.b.4}$$

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Physics 523, Quantum Field Theory II

Homework 8

Due Wednesday, 10th March 2004

Jacob Lewis Bourjaily

Renormalization of Pseudo-Scalar Yukawa Theory

Let us consider the theory generated by the Lagrangian

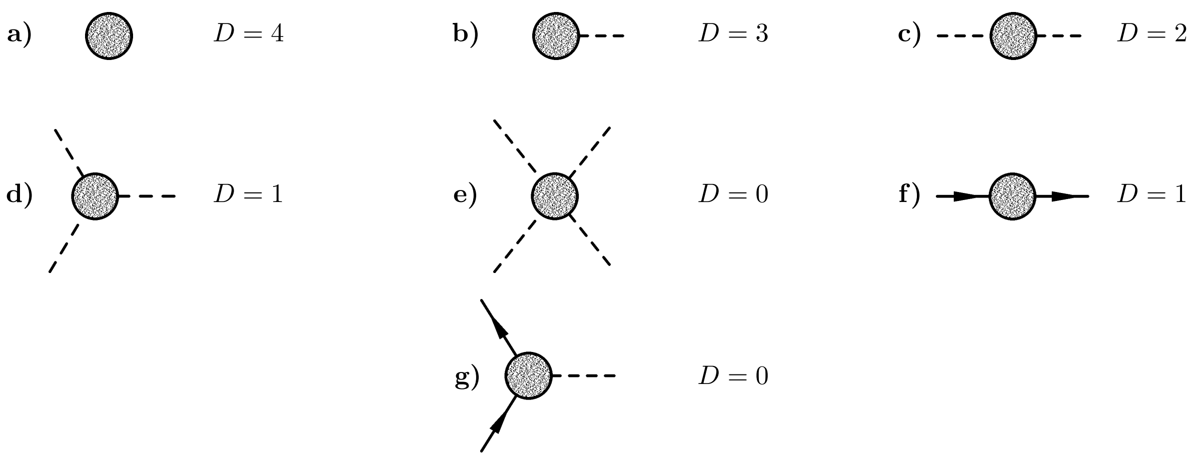
$$\mathscr{L} = \frac{1}{2}(\partial_\mu \phi_o)^2 - \frac{1}{2}m_{\phi_o}^2\phi_o^2 + \overline{\psi}_o(i\partial\!\!\!/ - m_{e_o})\psi_o - ig_o\overline{\psi}_o\gamma^5\psi_o\phi_o.$$

Superficially, this theory will diverge very similarly to quantum electrodynamics because the fields and the coupling constant have the same dimensions as in quantum electrodynamics. Therefore, we see that the superficial divergence is given by $D = 4L - 2P_\phi - P_e$ where L represents the number of loops and P_ϕ and P_e represent the number of pseudo-scalar and fermion propagator particles, respectively. Furthermore, we see that this can be reduced to

$$\boxed{D = 4 - N_\phi - \frac{3}{2}N_e,} \tag{a.1}$$

where N_ϕ and N_e represent the number of external pseudo-scalar and fermion lines, respectively.

We see that this implies that the following diagrams are superficially divergent:



Although vacuum energy is an extraordinarily interesting problem of physics, we will largely ignore diagram (a) which is quite divergent. We note that because the Lagrangian is invariant under parity transformations $\phi(t, \mathbf{x}) \to -\phi(t, -\mathbf{x})$ any diagram with an odd number of external ϕ's will give zero. In particular, the divergent diagrams (b) and (d) will be zero.

The first divergent diagram we will consider, (c), is clearly $\sim a_0\Lambda^2 + a_1p^2\log\Lambda$ where we note that the term proportional to p in the expansion vanishes by parity symmetry. Similarly, we naïvely suspect that the divergence of diagram (f) would be $\sim a_0\Lambda + p\!\!\!/ \log\Lambda$ but the term linear in Λ is reduced to $m_e \log\Lambda$ by the symmetry of the Lagrangian of chirality inversion of ψ together with $\phi \to -\phi$. The diagrams (e) and (g) are both $\sim \log\Lambda$. All together, there are six divergent constants in this theory.

We note that because the diagram (e) diverges, we must introduce a counterterm δ_λ which implies that our original Lagrangian should have included a term $\frac{\lambda}{4!}\phi^4$.

We define renormalized fields, $\phi_o \equiv Z_\phi^{1/2}\phi$ and $\psi_o \equiv Z_2^{1/2}\psi$, where Z_ϕ and Z_2 are as would be defined canonically. Using these our Lagrangian can be written as

$$\mathscr{L} = \frac{1}{2}Z_\phi(\partial_\mu\phi)^2 - \frac{1}{2}Z_\phi m_{\phi_o}^2\phi^2 - Z_2\overline{\psi}(i\partial\!\!\!/ - m_{e_o})\psi - -ig_oZ_2Z_\phi^{1/2}\overline{\psi}\gamma^5\psi\phi - \frac{\lambda}{4!}Z_\phi^2\phi^4.$$

Let us define the counterterms,

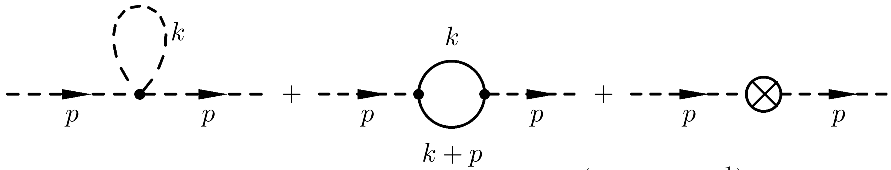
$$\delta_{m_\phi} \equiv Z_\phi m_{\phi_o}^2 - m_\phi^2, \quad \delta_{m_e} \equiv Z_2 m_{e_o} - m_e, \quad \delta_\phi \equiv Z_\phi - 1, \quad \delta_\lambda \equiv \lambda_o Z_\phi^2 - \lambda, \quad \delta_1 \equiv \frac{g_o}{g}Z_2Z_\phi^{1/2} - 1, \quad \delta_2 \equiv Z_2 - 1.$$

Therefore, we may write our renormalized Lagrangian

$$\begin{aligned}\mathscr{L} = &\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m_\phi^2\phi^2 + \overline{\psi}(i\partial\!\!\!/ - m_e)\psi - ig\overline{\psi}\gamma^5\psi\phi - \frac{\lambda}{4!}\phi^4 \\ &+ \frac{1}{2}\delta_\phi(\partial_\mu\phi)^2 - \frac{1}{2}\delta_{m_\phi}\phi^2 + \overline{\psi}(i\delta_2\partial\!\!\!/ - \delta_{m_e})\psi - ig\delta_1\overline{\psi}\gamma^5\psi\phi - \frac{\delta_\lambda}{4!}\phi^4.\end{aligned} \tag{a.4}$$

Let us compute the pseudo-scalar self-energy diagrams to the one-loop order, keeping only the divergent pieces. This corresponds to:

$$-iM^2(p^2) =$$



Using the 'canonical procedure' and dropping all but divergent pieces (linear in ϵ^{-1}) we see that

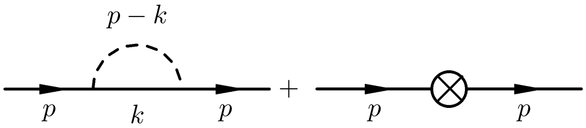
$$\begin{aligned}
-iM^2(p^2) &= -i\frac{\lambda}{2}\int\frac{d^dk}{(2\pi)^d}\frac{i}{k^2-m_\phi^2} - g^2\int\frac{d^dk}{(2\pi)^d}\mathrm{Tr}\left[\frac{\gamma^5 i(\not{k}+\not{p}+m_e)i\gamma^5(\not{k}+m_e)}{((k+p)^2-m_e^2)(k^2-m_e^2)}\right] + i(p^2\delta_\phi - \delta_{m_e}),\\
&= -i\frac{\lambda}{2}\frac{1}{(4\pi)^{d/2}}\frac{\Gamma\left(1-\frac{d}{2}\right)}{(m_\phi^2)^{1-d/2}} - 4g^2\int_0^1 dx\int\frac{d^dk}{(2\pi)^d}\frac{\ell^2 - x(1-x)p^2 - m_e^2}{(\ell^2-\Delta)^2} + i(p^2\delta_\phi - \delta_{m_2}),\\
&= -i\frac{\lambda}{2}\frac{1}{(4\pi)^{d/2}}\frac{m_\phi^2}{(1-d/2)}\frac{\Gamma\left(2-\frac{d}{2}\right)}{(m^2)^{2-d/2}} - 4g^2\int_0^1 dx\left[-\frac{i}{(4\pi)^{d/2}}\frac{d}{2}\frac{\Gamma\left(1-\frac{d}{2}\right)}{\Delta^{1-d/2}} + \frac{i}{(4\pi)^{d/2}}\frac{\Gamma\left(2-\frac{d}{2}\right)}{\Delta^{2-d/2}}\left(x(1-x)p^2+m_e^2\right)\right]\\
&\quad + i(p^2\delta_\phi - \delta_{m_2}),\\
&\sim i\frac{\lambda m_\phi^2}{32\pi^2}\frac{2}{\epsilon} - 8g^2\frac{i}{(4\pi)^2}\frac{2}{\epsilon}\int_0^1 dx\left(m_e^2 - x(1-x)p^2\right) + 4g^2\frac{i}{(4\pi)^2}\frac{2}{\epsilon}\int_0^1 dx\left(m_e^2 + x(1-x)p^2\right) + i(p^2\delta_\phi - \delta_{m_2}),\\
&= i\frac{\lambda m_\phi^2}{16\pi^2}\frac{1}{\epsilon} + i\frac{g^2}{4\pi^2}\frac{2}{\epsilon}\left(-2m_e^2 + \frac{2}{6}p^2 + \frac{1}{6}p^2 + m_e^2\right) + i(p^2\delta_\phi - \delta_{m_2}),\\
&= i\left(\frac{\lambda m_\phi^2}{16\pi^2} + \frac{g^2p^2}{4\pi^2} - \frac{g^2m_e^2}{2\pi^2}\right)\frac{1}{\epsilon} + i(p^2\delta_\phi - \delta_{m_2}).
\end{aligned}$$

Therefore, applying our renormalization conditions, we see that[1]

$$\boxed{\therefore\ \delta_{m_\phi} = \left(\frac{\lambda m_\phi^2}{16\pi^2} - \frac{g^2m_e^2}{2\pi^2}\right)\frac{1}{\epsilon}, \qquad \delta_\phi = -\left(\frac{g^2}{4\pi^2}\right)\frac{1}{\epsilon}.} \tag{b.1}$$

Similarly, let us compute the fermion self-energy diagrams to one-loop order, keeping only divergent parts. This corresponds to:

$$-i\Sigma_2^2(\not{p}) =$$



Again, using the 'canonical procedure' and dropping all but divergent pieces (linear in ϵ^{-1}) we see that

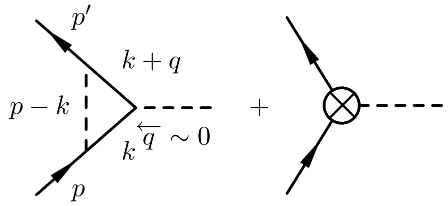
$$\begin{aligned}
-i\Sigma(\not{p}) &= g^2\int\frac{d^dk}{(2\pi)^d}\left[\gamma^5\frac{i}{((p-k)^2-m_\phi^2)}\frac{i(\not{k}+m_e)}{(k^2-m_e^2)}\gamma^5\right] + i(\not{p}\delta_2 - \delta_{m_e}),\\
&= -g^2\int\frac{d^dk}{(2\pi)^d}\frac{\not{k}-m_e}{(k^2-m_e^2)((p-k)^2-m_\phi^2)} + i(\not{p}\delta_2 - \delta_{m_e}),\\
&= -g^2\int_0^1 dz\int\frac{d^d\ell}{(2\pi)^d}\frac{\not{p}z - m_e}{(\ell^2-\Delta)^2} + i(\not{p}\delta_2 - \delta_{m_2}),\\
&\sim -i\frac{g^2}{(4\pi)^2}\frac{2}{\epsilon}\int_0^1 dz\,(\not{p}z - m_e) + i(\not{p}\delta_2 - \delta_{m_e}),\\
&= i\left(\frac{g^2\not{p}}{16\pi^2} - \frac{g^2m_e}{8\pi^2}\right)\frac{1}{\epsilon} + i\not{p}\delta_2 - i\delta_{m_e}.
\end{aligned}$$

Therefore, applying our renormalization conditions, we see that

$$\boxed{\therefore\ \delta_{m_e} = -\left(\frac{g^2m_e}{8\pi^2}\right)\frac{1}{\epsilon}, \qquad \delta_2 = -\left(\frac{g^2}{16\pi^2}\right)\frac{1}{\epsilon}.} \tag{b.2}$$

[1]For renormalization conditions and Feynman rules please see the Appendix.

Let us now compute the δ_1 counterterm by computing $\delta\Gamma^5(q=0)$ given by:

$\delta\Gamma^5(q=0) =$ 

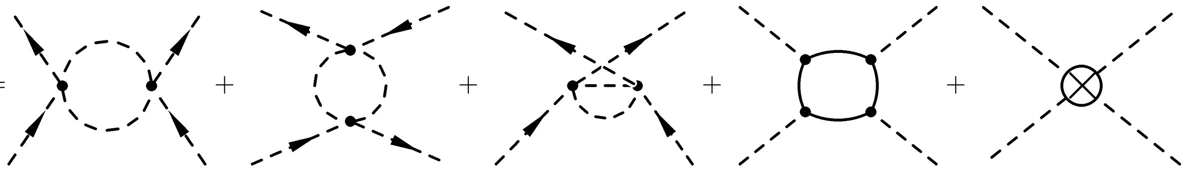
Again, using the 'canonical procedure' and dropping all but divergent pieces (linear in ϵ^{-1}) we see that

$$\begin{aligned}
\delta\Gamma^5(q=0) &= -ig^2\int\frac{d^dk}{(2\pi)^d}\frac{\gamma^5(\not{k}+m_e)\gamma^5(\not{k}+m_e)\gamma^5}{((p-k)^2-m_\phi^2)(k^2-m_e^2)(k^2-m^2)}+\delta_1\gamma^5,\\
&= ig^2\gamma^5\int\frac{d^dk}{(2\pi)^d}\frac{(\not{k}+m_e)(\not{k}-m_e)}{((p-k)^2-m_\phi^2)(k^2-m_e^2)(k^2-m^2)}+\delta_1\gamma^5,\\
&= ig^2\gamma^5\int_0^1 dz\int\frac{d^d\ell}{(2\pi)^d}\frac{\ell^2+(z^2-1)m_e^2}{(\ell^2-\Delta)^3}+\delta_1\gamma^5,\\
&= ig^2\gamma^5\int_0^1 dz(1-z)\left[\frac{i}{(4\pi)^2}\frac{d}{2}\frac{2}{\epsilon}\right]+\delta_1\gamma^5,\\
&= -\gamma^5\frac{g^2}{8\pi^2}\frac{1}{\epsilon}+\delta_1\gamma^5.
\end{aligned}$$

Therefore, applying our renormalization conditions, we see that

$$\boxed{\therefore\ \delta_1=\left(\frac{g^2}{8\pi^2}\right)\frac{1}{\epsilon}.} \tag{b.3}$$

Let us now compute the δ_λ counterterm by computing the one-loop correction to the standard ϕ^4 vertex. The five contributing diagrams are:

$i\mathscr{M} =$ 

We may save a bit of sweat by noting that the sum of the first four diagrams is identical to the analogous diagrams in ϕ^4-theory. The sum was computed fully both in class and in the text and give a divergent contribution of $\frac{3\lambda^2}{16\pi^2}\frac{1}{\epsilon}$ to δ_λ. Therefore, we are only burdened with the calculation of the remaining two. We see that, (note the combinatorial factor of 6)

$$\begin{aligned}
i\mathscr{M} &= i\frac{3\lambda^2}{16\pi^2}\frac{1}{\epsilon}-6g^4\int\frac{d^dk}{(2\pi)^d}\mathrm{Tr}\left[\frac{\gamma^5(\not{k}+m_e)\gamma^5(\not{k}-\not{p}_1+m_e)\gamma^5(\not{k}-\not{p}_1-\not{p}_2+m_e)\gamma^5(\not{k}-\not{p}_1-\not{p}_2+\not{p}_3+m_e)}{(k^2-m_e^2)((k-p_1)^2-m_e^2)((k-p_1-p_2)^2-m_e^2)((k-p_1-p_2+p_3)^2-m_e^2)}\right]-i\delta_\lambda,\\
&\underset{k\to\infty}{\simeq} i\frac{3\lambda^2}{16\pi^2}\frac{1}{\epsilon}-6g^4\int\frac{d^dk}{(2\pi)^d}\frac{\mathrm{Tr}\left[\gamma^5\not{k}\gamma^5\not{k}\gamma^5\not{k}\gamma^5\not{k}\right]}{(k^2-m_e^2)^4}-i\delta_\lambda,\\
&= i\frac{3\lambda^2}{16\pi^2}\frac{1}{\epsilon}-6g^4\int\frac{d^dk}{(2\pi)^d}\frac{4k^4}{(k^2-m_e^2)^4}-i\delta_\lambda,\\
&= i\frac{3\lambda^2}{16\pi^2}\frac{1}{\epsilon}-24g^2\frac{i}{(4\pi)^{d/2}}\frac{d(d+2)}{4}\frac{\Gamma\left(2-\frac{d}{2}\right)}{6\Delta^{2-d/2}}-i\delta_\lambda,\\
&= i\frac{3\lambda^2}{16\pi^2}\frac{1}{\epsilon}-i\frac{3g^4}{\pi^2}\frac{1}{\epsilon}-i\delta_\lambda.
\end{aligned}$$

Therefore, applying our renormalization conditions, we see that

$$\boxed{\therefore\ \delta_\lambda=\left(\frac{3\lambda^2}{16\pi^2}-\frac{3g^4}{\pi^2}\right)\frac{1}{\epsilon}.} \tag{b.4}$$

Appendix

Feynman Rules and Renormalization Conditions

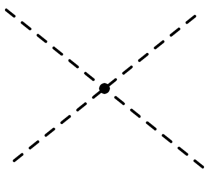
Given the Lagrangian for pseudo-scalar Yukawa theory,

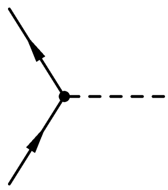
$$\mathscr{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m_\phi^2\phi^2 + \overline{\psi}(i\not\partial - m_e)\psi - ig\overline{\psi}\gamma^5\psi\phi - \frac{\lambda}{4!}\phi^4$$
$$+\frac{1}{2}\delta_\phi(\partial_\mu\phi)^2 - \frac{1}{2}\delta_{m_\phi}\phi^2 + \overline{\psi}(i\delta_2\not\partial - \delta_{m_e})\psi - ig\delta_1\overline{\psi}\gamma^5\psi\phi - \frac{\delta_\lambda}{4!}\phi^4,$$

we can derive the renormalized Feynman rules.

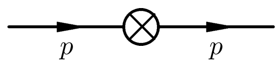
$= \frac{i}{p^2 - m_{\phi}{}^2 + i\epsilon}$

$= \frac{i}{\not p - m_e + i\epsilon}$

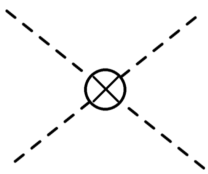

$= -i\lambda$

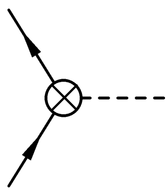

$= g\gamma^5$


$= i(p^2\delta_\phi - \delta_{m_\phi})$

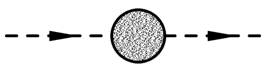


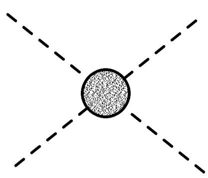
$= i(\not p\delta_2 - \delta m_e)$


$= -i\delta_\lambda$


$= g\delta_1\gamma^5$

To derive the counter terms explicitly, it is necessary to offer a convention of renormalization conditions. Above, we have used the conditions:


$= \frac{i}{p^2 - m_{\phi}{}^2 + i\epsilon}$ with pole $= 1$.


$= -i\lambda$ at $s = 4m^2, t = u = 0$.

$$\Sigma(\not p = m) = 0.$$
$$\left.\frac{d\Sigma(\not p)}{d\not p}\right|_{\not p = m} = 0.$$
$$g\Gamma^5(q = 0) = g\gamma^5.$$

Physics 523, Quantum Field Theory II

Homework 9

Due Wednesday, 17th March 2004

Jacob Lewis Bourjaily

β-Functions in Pseudo-Scalar Yukawa Theory

Let us consider the massless pseudo-scalar Yukawa theory governed by the renormalized Lagrangian,

$$\begin{aligned}\mathscr{L} &= \frac{1}{2}(\partial_\mu\phi)^2 + \overline{\psi} i \partial\!\!\!/ \psi - ig\overline{\psi}\gamma^5\psi\phi - \frac{\lambda}{4!}\phi^4 \\ &+ \frac{1}{2}\delta_\phi(\partial_\mu\phi)^2\overline{\psi} i\delta_\psi \partial\!\!\!/ \psi - ig\delta_g\overline{\psi}\gamma^5\psi\phi - \frac{\delta_\lambda}{4!}\phi^4.\end{aligned}$$

In homework 8, we calculated the divergent parts of the renormalization counterterms $\delta_\phi, \delta_\psi, \delta_g$, and δ_λ to 1-loop order. These were shown to be

$$\delta_\phi = -\frac{g^2}{8\pi^2}\log\frac{\Lambda^2}{M^2}, \qquad \delta_\psi = -\frac{g^2}{32\pi^2}\log\frac{\Lambda^2}{M^2};$$

$$\delta_\lambda = \left(\frac{3\lambda^2}{32\pi^2} - \frac{3g^4}{2\pi^2}\right)\log\frac{\Lambda^2}{M^2}, \qquad \delta_g = \frac{g^2}{16\pi^2}\log\frac{\Lambda^2}{M^2}.$$

Using the definitions of B_i and A_i in Peskin and Schroeder, these imply that

$$A_\phi = -\gamma_\phi = -\frac{g^2}{8\pi^2}, \qquad A_\psi = -\gamma_\psi = -\frac{g^2}{32\pi^2};$$

$$B_\lambda = \frac{3g^4}{2\pi^2} - \frac{3\lambda^2}{32\pi^2} \qquad B_g = -\frac{g^2}{16\pi^2}.$$

Therefore, we see that

$$\begin{aligned}\beta_g &= -2gB_g - 2gA_\psi - gA_\phi = 2g\frac{g^2}{16\pi^2} + 2g\frac{g^2}{32\pi^2} + g\frac{g^2}{8\pi^2} = \frac{5g^3}{16\pi^2};\\ \beta_\lambda &= -2B_\lambda - 4\lambda A_\phi = 2\left(\frac{3\lambda^2}{32\pi^2} - \frac{3g^4}{2\pi^2}\right) + 4\lambda\frac{g^2}{8\pi^2} = \frac{3\lambda^2 + 8\lambda g^2 - 48g^4}{16\pi^2}\end{aligned}$$

While it was supposedly unnecessary, the running couplings were computed to be[1],

$$\begin{aligned}\overline{g}(p) &= \sqrt{\frac{16\pi^2}{1 - 10\log p/M}};\\ \overline{\lambda}(p) = \overline{\lambda} &= \frac{\overline{g}^2}{3}\left(1 + \sqrt{145}\frac{-\frac{4\sqrt{145}+149}{141} + \overline{g}^{2\sqrt{145}/5}}{-\frac{4\sqrt{145}+149}{141} - \overline{g}^{2\sqrt{145}/5}}\right).\end{aligned}$$

Notice that both $\overline{g}$ and $\overline{\lambda}$ generally become weaker at large distances because for typical values of g, λ we see that β_g and β_λ are both positive. However, if $\lambda << g$ then β_λ will be negative and so $\overline{\lambda}$ will grow stronger at larger distances. Near small values of g and λ the theory shows interesting interplay between g and λ. Also interesting is the characteristic Landau pole in $\overline{\lambda}$ suggesting that we should not trust this theory at too large a scale.

Below is a graph of $\overline{g}$ versus $-\overline{\lambda}$ indicating the direction of Renormalization Group flow as the interaction distance grows larger.

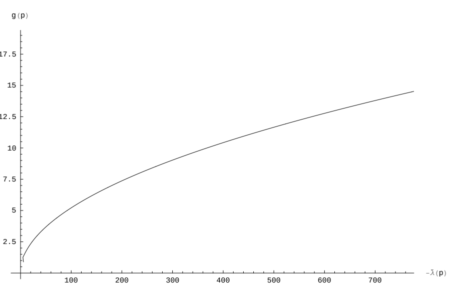


Figure 1. Renormalization Group Flow as a funciton of scale. Arrow indicates flow in the direction of larger distances. For this plot, M was taken to be 10^4.

[1]See appendix.

Minimal Subtraction

Let us define the β-function as it appears in dimensional regularization as

$$\beta(\lambda,\epsilon) = M\frac{d}{dM}\lambda\bigg|_{\lambda_0,\epsilon},$$

where it is understood that $\beta(\lambda) = \lim_{\epsilon\to 0}\beta(\lambda,\epsilon)$. We notice that the bare coupling is given by $\lambda_0 = M^\epsilon Z_\lambda(\lambda,\epsilon)\lambda$ where Z_λ is given by an expansion series in ϵ,

$$Z_\lambda(\lambda,\epsilon) = 1 + \sum_{\nu=1}\frac{a_\nu(\lambda)}{\epsilon^\nu}.$$

We are to demonstrate the following.

a) Let us show that Z_λ satisfies the identity $(\beta(\lambda,\epsilon)+\epsilon\lambda)Z_\lambda + \beta(\lambda\epsilon)\lambda\frac{dZ_\lambda}{d\lambda} = 0$.

proof: Noting the general properties of differentiation from elementary analysis, we will proceed by direct computation.

$$\begin{aligned}(\beta(\lambda,\epsilon)+\epsilon\lambda)\,Z_\lambda + \beta(\lambda,\epsilon)\lambda\frac{dZ_\lambda}{d\lambda} &= \beta(\lambda,\epsilon)Z_\lambda + \epsilon\lambda Z_\lambda + \beta(\lambda,\epsilon)\frac{d(Z_\lambda\lambda)}{d\lambda} - \beta(\lambda,\epsilon)Z_\lambda,\\ &= \epsilon\lambda Z_\lambda + M\frac{d\lambda}{dM}\bigg|_{\lambda_0,\epsilon}\frac{d(\lambda_0 M^{-\epsilon})}{d\lambda},\\ &= \epsilon\lambda Z_\lambda - \epsilon M\lambda_0 M^{-\epsilon-1},\\ &= \epsilon\lambda Z_\lambda - \epsilon M^{1+\epsilon}M^{-\epsilon-1}Z_\lambda\lambda,\\ &= 0.\end{aligned}$$

$$\boxed{\therefore (\beta(\lambda,\epsilon)+\epsilon\lambda)Z_\lambda + \beta(\lambda\epsilon)\lambda\frac{dZ_\lambda}{d\lambda} = 0.}$$

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b) Let us show that $\beta(\lambda,\epsilon) = -\epsilon\lambda + \beta(\lambda)$.

proof: We have demonstrated in part (a) above that $(\beta(\lambda,\epsilon)+\epsilon\lambda)Z_\lambda + \beta(\lambda\epsilon)\lambda\frac{dZ_\lambda}{d\lambda} = 0$. Dividing this equation by Z_λ and rearranging terms and expanding in Z_λ, we obtain

$$\begin{aligned}\beta(\lambda,\epsilon)+\epsilon\lambda &= -\beta(\lambda,\epsilon)\frac{\lambda}{Z_\lambda}\frac{dZ_\lambda}{d\lambda},\\ &= -\beta(\lambda,\epsilon)\frac{\lambda}{Z_\lambda}\left(\frac{1}{\epsilon}\frac{da_1}{d\lambda} + \frac{1}{\epsilon^2}\frac{da_2}{d\lambda} + \cdots\right),\\ &= -\beta(\lambda,\epsilon)\lambda\left(\frac{1}{\epsilon}\frac{da_1}{d\lambda} + \frac{1}{\epsilon^2}\frac{da_2}{d\lambda} + \cdots\right)\left(1 - \frac{a_1}{\epsilon} + \cdots\right).\end{aligned}$$

Now, we know that $\beta(\lambda,\epsilon)$ must be regular in ϵ as $\epsilon\to 0$ and so we may expand it as a (terminating)[2] power series $\beta(\lambda,\epsilon) = \beta_0 + \beta_1\epsilon + \beta_2\epsilon^2 + \cdots + \beta_n\epsilon^n$. We notice that $\beta(\lambda) = \beta_0$ in this notation. Let us consider the limit of $\epsilon\to\infty$.

For any $n > 0$, we see that the order of the polynomial on the left hand side has degree n whereas the polynomial on the left hand side has degree $n-1$ because as $\epsilon\to\infty$, the equation becomes $\beta_n\epsilon^n = -\beta_n\epsilon^n\lambda\frac{1}{\epsilon}\frac{da_1}{d\lambda}$. But this is a contradiction. $\rightarrow\!\leftarrow$

Therefore, both the right and left hand sides must have degree less than or equal to 0. Furthermore, because the left hand side is $\beta(\lambda,\epsilon)+\epsilon\lambda = \beta_0 + \beta_1\epsilon + \epsilon\lambda$ must have degree zero, we see that $\beta_1 = -\epsilon$.

So, expanding $\beta(\lambda,\epsilon)$ as a power series of ϵ, we obtain,

$$\boxed{\therefore \beta(\lambda,\epsilon) = -\epsilon\lambda + \beta(\lambda).}$$

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[2]Professor Larsen does note believe this to be necessary. However, we have been unable to demonstrate the required identity without assuming a terminating power series.

c.i) Let us show that $\beta(\lambda) = \lambda^2 \frac{da_1}{d\lambda}$.

proof: By rewriting the identity obtained from part (a) above and expanding in Z_λ we see that

$$(\beta(\lambda,\epsilon) + \epsilon\lambda)\, Z_\lambda = -\beta(\lambda,\epsilon)\lambda \frac{dZ_\lambda}{d\lambda},$$
$$(\beta(\lambda,\epsilon) + \epsilon\lambda)\left(1 + \frac{a_1}{\epsilon} + \cdots\right) = -\beta(\lambda,\epsilon)\lambda\left(\frac{1}{\epsilon}\frac{da_1}{d\lambda} + \cdots\right).$$

We see that because there is no term on the right hand side of order ϵ^0, it must be that $\beta(\lambda,\epsilon) + \lambda a_1 = 0$ which implies that $\beta(\lambda,\epsilon) = -\lambda a_1$. Furthermore, by equating the coefficients of $\frac{1}{\epsilon^n}$, we have in general that $\beta(\lambda,\epsilon)a_n + \lambda a_{n+1} = -\beta(\lambda,\epsilon)\lambda\frac{da_n}{d\lambda}$. By rearranging terms and using noticing the chain rule of differentiation, we see that this implies that

$$\lambda a_{n+1} = -\beta(\lambda,\epsilon)\left(\lambda\frac{da_n}{d\lambda} + a_n\right) = -\beta(\lambda,\epsilon)\frac{d(\lambda a_n)}{d\lambda}.$$

This fact will be important to the proof immediately below.

Now, by the result of part (b) above, we know that

$$\beta(\lambda)Z_\lambda = (\beta(\lambda,\epsilon) + \epsilon\lambda)Z_\lambda = -\beta(\lambda,\epsilon)\lambda\frac{dZ_\lambda}{d\lambda},$$
$$\beta(\lambda)\left(1 + \frac{a_1}{\epsilon} + \cdots\right) = (\beta(\lambda,\epsilon) + \epsilon\lambda)Z_\lambda = -\beta(\lambda,\epsilon)\lambda\left(\frac{1}{\epsilon}\frac{da_1}{d\lambda} + \cdots\right).$$

Equating the coefficients of terms of order $\frac{1}{\epsilon}$ on the far left and right sides, we see that

$$\beta(\lambda)a_1 = -\beta(\lambda,\epsilon)\lambda\frac{da_1}{d\lambda}.$$

Now, using our result from before that $\beta(\lambda,\epsilon) = -\lambda a_1$, we see directly that

$$\boxed{\therefore \beta(\lambda) = \lambda^2\frac{da_1}{d\lambda}.}$$

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c.ii) Let us show that $\beta(\lambda)\frac{d(\lambda a_\nu)}{d\lambda} = \lambda^2\frac{da_{\nu+1}}{d\lambda}$.

proof: By our result in part (b) above, we have that

$$\beta(\lambda) = (\beta(\lambda,\epsilon) + \epsilon\lambda),$$
$$\therefore \beta(\lambda)\frac{d(Z_\lambda\lambda)}{d\lambda} = (\beta(\lambda,\epsilon) + \epsilon\lambda)\frac{d(Z_\lambda\lambda)}{d\lambda},$$
$$\beta(\lambda)\left(1 + \frac{1}{\epsilon}\frac{d(\lambda a_1)}{d\lambda} + \cdots\right) = (\beta(\lambda,\epsilon) + \epsilon\lambda)\left(1 + \frac{1}{\epsilon}\frac{d(\lambda a_1)}{d\lambda} + \cdots\right).$$

Equating the coefficients of $\frac{1}{\epsilon^\nu}$ on both sides, we see that by using the identities shown above,

$$\begin{aligned}\beta(\lambda)\frac{d(\lambda a_\nu)}{d\lambda} &= \beta(\lambda,\epsilon)\frac{d(\lambda a_\nu)}{d\lambda} + \lambda\frac{d(\lambda a_{\nu+1})}{d\lambda},\\ &= \beta(\lambda,\epsilon)\frac{d(\lambda a_\nu)}{d\lambda} + \lambda^2\frac{da_{\nu+1}}{d\lambda} + \lambda a_{\nu+1},\\ &= \beta(\lambda,\epsilon)\frac{d(\lambda a_\nu)}{d\lambda} + \lambda^2\frac{da_{\nu+1}}{d\lambda} - \beta(\lambda,\epsilon)\frac{d(\lambda a_\nu)}{d\lambda},\\ &= \lambda^2\frac{da_{\nu+1}}{d\lambda}.\end{aligned}$$

So we see in general that

$$\boxed{\therefore \beta(\lambda)\frac{d(\lambda a_\nu)}{d\lambda} = \lambda^2\frac{da_{\nu+1}}{d\lambda}.}$$

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In the minimal subtraction scheme, we define the mass renormalization by $m_0^2 = m^2 Z_m$ where

$$Z_m = 1 + \sum_{\nu=1} \frac{b_\nu}{\epsilon^\nu}.$$

Similarly, we will define the associated β-function $\beta_m(\lambda) = m\gamma_m(\lambda)$ which is given by

$$\beta_m(\lambda) = M \frac{dm}{dM}\bigg|_{m_0,\epsilon}.$$

d.i) Let us show that $\gamma_m(\lambda) = \frac{\lambda}{2}\frac{db_1}{d\lambda}$.

proof: Because m_0^2 is a constant, we know that $\frac{dm_0^2}{dM} = 0$. Therefore, writing $m_0^2 = m^2 Z_m$ we see that this implies

$$\begin{aligned}
\frac{dm_0^2}{dM} = 0 &= 2Z_m m \frac{dm}{dM} + m^2 \frac{dZ_m}{dM}, \\
&= 2Z_m m \frac{\beta_m(\lambda)}{M} + m^2 \frac{dZ_m}{d\lambda}\frac{d\lambda}{dM} = 0; \\
\therefore 0 &= 2Z_m \beta_m(\lambda) + mM \frac{d\lambda}{dM}\frac{dZ_m}{d\lambda}; \\
\therefore 2\beta_m(\lambda) Z_m &= -m\beta(\lambda,\epsilon)\frac{dZ_m}{d\lambda}, \\
2\beta_m(\lambda)\left(1 + \frac{b_1}{\epsilon} + \cdots\right) &= -m\beta(\lambda,\epsilon)\left(\frac{1}{\epsilon}\frac{db_1}{d\lambda} + \cdots\right), \\
2\beta_m(\lambda)\left(1 + \frac{b_1}{\epsilon} + \cdots\right) &= -m\left(\beta(\lambda) - \epsilon\lambda\right)\left(\frac{1}{\epsilon}\frac{db_1}{d\lambda} + \cdots\right),
\end{aligned}$$

We see that the coefficient of the ϵ^0 term on the left hand side is $2\beta_m(\lambda)$ and on the right hand side it is $m\lambda\frac{db_1}{d\lambda}$. Therefore, because these terms must be equal, we see that

$$\beta_m(\lambda) = m\frac{\lambda}{2}\frac{db_1}{d\lambda},$$

$$\boxed{\therefore \gamma_m(\lambda) = \frac{\lambda}{2}\frac{db_1}{d\lambda}.}$$

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d.ii) Let us prove that $\lambda\frac{db_{\nu+1}}{d\lambda} = 2\gamma_m(\lambda)b_\nu + \beta(\lambda)\frac{db_\nu}{d\lambda}$.

proof: Continuing our work from part (d.i) above, we have that

$$2\beta_m(\lambda)\left(1 + \frac{b_1}{\epsilon} + \cdots\right) = -m\left(\beta(\lambda) - \epsilon\lambda\right)\left(\frac{1}{\epsilon}\frac{db_1}{d\lambda} + \cdots\right).$$

It must be that the coefficients of $\frac{1}{\epsilon^\nu}$ are equal on both sides. Therefore, we see that

$$\begin{aligned}
2\beta_m(\lambda)b_\nu &= -m\beta(\lambda)\frac{db_\nu}{d\lambda} + m\lambda\frac{db_{\nu+1}}{d\lambda}, \\
2m\gamma_m(\lambda)b_\nu &= -m\beta(\lambda)\frac{db_\nu}{d\lambda} + m\lambda\frac{db_{\nu+1}}{d\lambda}, \\
\therefore 2\gamma_m(\lambda)b_\nu &= -\beta(\lambda)\frac{db_\nu}{d\lambda} + \lambda\frac{db_{\nu+1}}{d\lambda}.
\end{aligned}$$

Rearranging terms, we see that

$$\boxed{\therefore \lambda\frac{db_{\nu+1}}{d\lambda} = 2\gamma_m(\lambda)b_\nu + \beta(\lambda)\frac{db_\nu}{d\lambda}.}$$

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Appendix

Calculation of the Running Couplings $\overline{g}$ and $\overline{\lambda}$

Let us now solve for the flow of the coupling constants g, λ. We have in general that solutions to the Callan-Symanzik equation will satisfy

$$\frac{d\overline{g}}{d\log p/M} = \beta_g = \frac{5g^3}{16\pi^2} + \mathcal{O}(g^5).$$

This is an ordinary differential equation. We see that

$$-\frac{1}{2}\frac{1}{\overline{g}^2} = \frac{5}{16\pi^2}\log p/M + C,$$

and so

$$\therefore \overline{g}^2(p) = -\frac{8\pi^2}{5\log p/M + C}.$$

The constant C is found so that $g(p = M) = 1$.[3] This yields $C = -1/2$.

To find the flow of λ, however, it will be convenient to introduce a new variable $\eta \equiv \lambda/g^2$. We must then solve the equation

$$\frac{d\overline{\eta}}{d\log p/M} = \frac{\beta_\lambda}{g^2} - 2\frac{\lambda\beta_g}{g^3} = \frac{\left(3\eta^2 - 2\eta - 48\right)g^2}{16\pi^2} + \mathcal{O}(g^4).$$

This is again a simple ordinary differential equation. We see that this implies

$$\int\frac{d\overline{\eta}}{3\eta^2 - 2\eta - 48} = \int\frac{g^2}{16\pi^2}d\log p/M.$$

Note that from our work above, $\frac{g^2}{16\pi^2}d\log p/M = \frac{g^2}{16\pi^2}d\left(-\frac{8\pi^2}{5g^2}\right) = \frac{1}{5g}dg$. Therefore,

$$\int\frac{d\overline{\eta}}{3\eta^2 - 2\eta - 48} = \int\frac{1}{5g}dg.$$

And so,

$$\log\left(\frac{3\overline{\eta} - \sqrt{145} - 1}{3\overline{\eta} + \sqrt{145} - 1}\right) = \frac{2\sqrt{145}}{5}\log g + C.$$

Solving this equation in terms of η, we see that we have

$$\begin{aligned}
\overline{\eta} &= \frac{Cg^{2\sqrt{145}/5}\left(\sqrt{145} - 1\right) + \sqrt{145} + 1}{3 - 3Cg^{2\sqrt{145}/5}}, \\
&= \frac{1 - Cg^{2\sqrt{145}/5}}{3 - 3Cg^{2\sqrt{145}/5}} + \frac{Cg^{2\sqrt{145}/5}\sqrt{145} + \sqrt{145}}{3 - 3Cg^{2\sqrt{145}/5}}, \\
&= \frac{1}{3}\left(1 + \sqrt{145}\frac{C + g^{2\sqrt{145}/5}}{C - g^{2\sqrt{145}/5}}\right).
\end{aligned}$$

$$\therefore \overline{\lambda} = \frac{g^2}{3}\left(1 + \sqrt{145}\frac{C + g^{2\sqrt{145}/5}}{C - g^{2\sqrt{145}/5}}\right).$$

As before, the constant term C is found by requiring that $\overline{\lambda}(p = M) = 1$. The constant is then $C = -\frac{4\sqrt{145}+149}{141}$.

[3] It can be argued that this is a poor choice of C because it requires the reference scale to be non-perturbative. Nevertheless, it is not a free parameter.

Physics 523, Quantum Field Theory II
Homework 10
Due Wednesday, 24$^{\text{th}}$ March 2004

Jacob Lewis Bourjaily

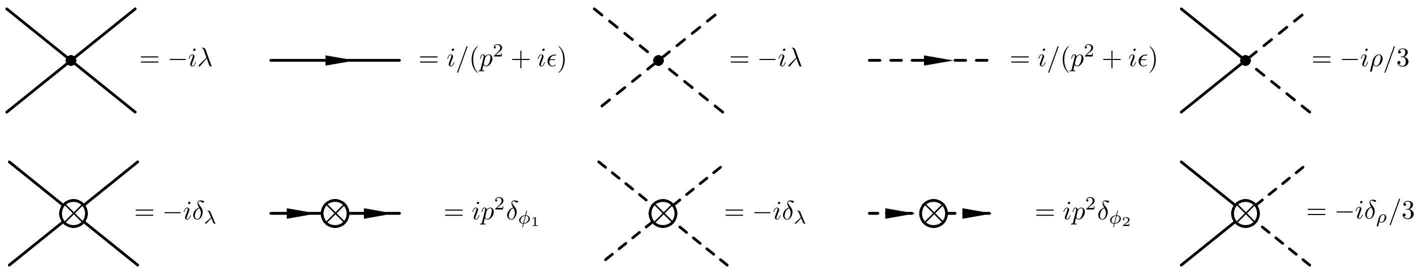
Asymptotic Symmetry

Let us consider the theory generated by the Lagrangian,

$$\mathscr{L} = \frac{1}{2}\left((\partial_\mu\phi_1)^2 + (\partial_\mu\phi_2)^2\right) - \frac{\lambda}{4!}\left(\phi_1^4 + \phi_2^4\right) - \frac{2\rho}{4!}\left(\phi_1^2\phi_2^2\right).$$

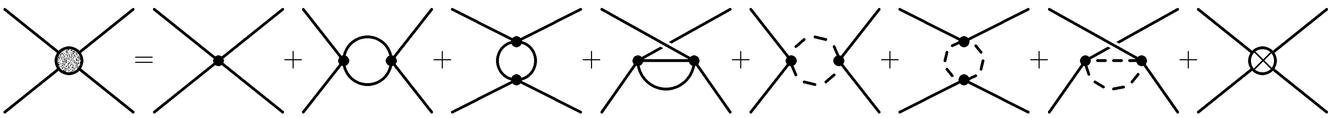
From this Lagrangian we may compute the Feynman rules. We notice that while the ϕ_i^4 interaction has a symmetry of 4! to cancel the denominator, there is only a symmetry of 4 associated with the $\phi_1^2\phi_2^2$ vertex and therefore the vertex factor is $-i4\cdot\frac{2\rho}{4!} = -i\frac{\rho}{3}$.

After we have renormalized with canonical renormalization conditions, the Feynman rules are: [1]



Let us now compute the β-functions for the couplings λ and ρ. To do this, we require the renormalization counter-terms δ_λ and δ_ρ.

To the one-loop order, we can find δ_λ by computing,



$$= -i\lambda + (-i\lambda)^2\left[V(t) + V(s) + V(u)\right] + (-i\frac{\rho}{3})^3\left[V(t) + V(s) + V(u)\right] - i\delta_\lambda,$$
$$= -i\lambda - \left(\lambda^2 + \frac{\rho^2}{9}\right)\left[V(t) + V(s) + V(u)\right] - i\delta_\lambda.$$

Now, we notice that the integral $V(k)$ is identical in all diagrams. In fact, every one-loop diagram we will concern ourselves with give the same loop integral $V(k)$. Let us compute the divergent piece of $V(k)$. Noticing the symmetry factor of $\frac{1}{2}$ and recalling our early results of dimensional regularization,

$$V(k) = \frac{1}{2}\int\frac{d^dk}{(4\pi)^d}\frac{i}{(k^2 + i\epsilon)}\frac{i}{((p-k)^2 + i\epsilon)},$$
$$= -\frac{1}{2}\int_0^1 dx\int\frac{d^d\ell}{(4\pi)^d}\frac{1}{(\ell^2 - \Delta)^2},$$
$$= -\frac{1}{2}\int_0^1 dx\frac{i}{(4\pi)^{d/2}}\frac{\Gamma\left(2 - \frac{d}{2}\right)}{\Delta^{2-d/2}},$$
$$\underset{d\to 4}{\sim} -\frac{i}{32\pi^2}\frac{2}{\epsilon} \to -\frac{i}{32\pi^2}\log\frac{\Lambda^2}{M^2}.$$

Therefore, applying the canonical renormalization conditions, we see that

$$\delta_\lambda = \frac{3}{32\pi^2}\left[\lambda^2 + (\rho/3)^2\right]\log\frac{\Lambda^2}{M^2}.$$

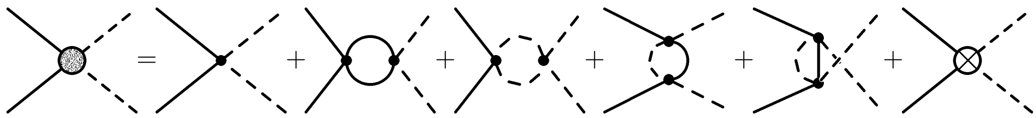
Because there are no divergent self-energy diagrams in this theory to one-loop order[2], we have that the β-function for λ is given precisely by twice the coefficient of the log divergence in δ_λ.

$$\boxed{\therefore \beta_\lambda = \frac{3}{16\pi^2}\left[\lambda^2 + (\rho/3)^2\right].} \qquad (1.b.1)$$

[1] Notice that we have used ——➤—— to represent the field ϕ_1 and we have used - - -➤- - - to represent the field ϕ_2.

[2] It is clear that the ϕ^4 interaction does not itself offer any self-energy divergences to one-loop order. Furthermore, we see that the $\phi_1^2\phi_2^2$ interaction's contribution to self-energy also involves a loop independent of external momentum and therefore will not diverge.

Similar to our computation above, to find β_ρ we must compute the renormalization counter-term δ_ρ. To the one-loop order, we can find δ_ρ by computing,



We notice that the symmetry factor of 2, included in our evaluation of the function $V(k)$, should not be included for the penultimate and antepenultimate diagrams because distinct fields run in the loop. Therefore, the loop integral for each of those two diagrams will contribute $2V(k)$ to to the total amplitude. Noting this subtlety, we find that

$$i\mathscr{M} = -i(\rho/3) + (-i\lambda)(-i\rho/3)\left[V(t) + V(t)\right] + (-i\rho/3)^2\left[2V(u) + 2V(s)\right] - i\delta_\rho/3.$$

Recall that we have already computed the divergence of the function $V(k)$ and noted that it was independent of k. Therefore,

$$\begin{aligned} i\delta_\rho/3 &= (-i\lambda)(-i\rho/3)\left[V(t) + V(t)\right] + (-i\rho/3)^2\left[2V(u) + 2V(s)\right], \\ &= -\lambda\rho/3\frac{-i}{16\pi^2}\log\frac{\Lambda^2}{M^2} - (\rho/3)^2\frac{-i}{8\pi^2}\log\frac{\Lambda^2}{M^2}, \\ \therefore \delta_\rho &= \frac{1}{16\pi^2}\left[\lambda\rho + 2\rho^2/3\right]\log\frac{\Lambda^2}{M^2}. \end{aligned}$$

Because there are no divergent self-energy diagrams in this theory to one-loop order, we have that the β-function for ρ is given precisely by twice the coefficient of the log divergence in δ_ρ.

$$\boxed{\therefore \beta_\rho = \frac{1}{8\pi^2}\left[\lambda\rho + 2\rho^2/3\right].} \tag{1.b.2}$$

Let us now consider the β-function associated with the ration λ/ρ. Using the chain rule for differentiation and the definition of the general β-function, we see that

$$\begin{aligned} \beta_{\lambda/\rho} = \frac{1}{\rho^2}\left[\beta_\lambda\rho - \beta_\rho\lambda\right] &= \frac{1}{\rho^2}\left[\frac{3\lambda^2\rho}{16\pi^2} + \frac{\rho^3}{48\pi^2} - \frac{\lambda^2\rho}{8\pi^2} - \frac{\rho^2\lambda}{12\pi^2}\right], \\ &= \frac{(\lambda/\rho)^2\rho}{16\pi^2} + \frac{\rho}{48\pi^2} - \frac{(\lambda/\rho)}{12\pi^2}, \\ &= \frac{\rho}{48\pi^2}\left[3(\lambda/\rho)^2 - 4(\lambda/\rho) + 1\right], \end{aligned}$$

$$\boxed{\therefore \beta_{\lambda/\rho} = \frac{\rho}{48\pi^2}\left(3\lambda/\rho - 1\right)\left(\lambda/\rho - 1\right).} \tag{1.c.1}$$

We see immediately that the two roots of $\beta_{\lambda/r}$ occur when $\lambda/\rho = 1, \frac{1}{3}$ and because the second derivative of $\beta_{\lambda/r}$ is $6 > 0$, we know that $\beta_{\lambda/\rho} < 0$ for $\lambda/\rho \in (\frac{1}{3}, 1)$ and $\beta_{\lambda/r} > 0$ for $\lambda/\rho > 1$. Therefore, for all $\lambda/\rho > \frac{1}{3}$, λ/ρ will flow to $\lambda/\rho = 1$. See Figure 1 below.

Therefore at large distances the couplings will flow to $\lambda = \rho$. This introduces a continuous $O(2)$ symmetry into the theory. To see this, let us define $\varphi \equiv \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$. In this notation, the Lagrangian simply reads

$$\boxed{\mathscr{L} = \frac{1}{2}(\partial_\mu\varphi)^2 - \frac{\lambda}{4!}\varphi^4.} \tag{1.e.1}$$

This Lagrangian is clearly invariant to $O(2)$ transformations which correspond to changing the phase of φ.

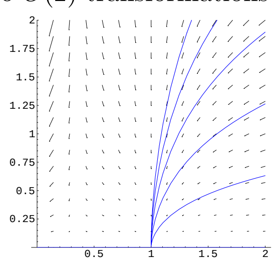


FIGURE 1. Renormalization Group Flow as a function of scale. Arrows show $p \to 0$ flow.

Asymptotic Freedom

Let us consider a theory with a coupling constant g such that

$$\beta(g) = -\frac{\beta_1 g^3}{16\pi^2} \qquad \text{and} \qquad \gamma(g) = \frac{\gamma_1 g^3}{16\pi^2},$$

for some positive constants β_1, γ_1.

The renormalized correlation functions satisfy the Callan-Symanzik equations which, for the amputated correlators, take the form

$$\left[M\frac{d}{dM} + \beta(g)\frac{\partial}{\partial g} - n\gamma(g)\right]\Gamma_R^{(n)}(p_i/m, g) = 0.$$

If we take all the momenta to be equal for simplicity, then the solutions to the Callan-Symanzik equations take the form

$$\Gamma_R^{(n)}(p/M, g) = \Gamma^{(n)}(\overline{g}(p/M))\exp\left(-4\int_M^p d\log(p'/M)\gamma(\overline{g}(p';g))\right).$$

Let us compute the running coupling $\overline{g}(p/M)$. By the Callan-Symanzik equations, we see that

$$\frac{d\overline{g}}{d\log(p/M)} = -\frac{\beta_1\overline{g}^3}{16\pi^2} \Longrightarrow \int_g^{\overline{g}}\frac{d\overline{g}}{\overline{g}^3} = -\frac{\beta_1}{16\pi^2}\int d\log(p/M),$$
$$\Longrightarrow -\frac{1}{2}\left(\frac{1}{\overline{g}^2} - \frac{1}{g^2}\right) = -\frac{\beta_1}{16\pi^2}\log(p/M),$$

$$\boxed{\therefore \overline{g}^2 = \frac{g^2}{1 + g^2\frac{\beta_1}{8\pi^2}\log(p/M)}.} \tag{2.b.1}$$

Therefore, we see immediately that when $p/M \to \infty$, 1 becomes insignificant in the denominator of $\overline{g}^2$ and so $\overline{g}$ becomes independent of g. We see that

$$\boxed{\therefore \overline{g}^2 \underset{p\to\infty}{\approx} \frac{8\pi^2}{\beta_1\log(p/M)}.} \tag{2.b.2}$$

Furthermore, we notice that this approximation can be trusted because nonperturbative effects become weaker at higher energy scales in an asymptotically free theory.

Let us now compute the dependence of the four-point vertex on momentum as $p/M \to \infty$. We assume that, to the lowest order, $\Gamma_R^{(4)} = g^2$. We cited the general solution to the (amputated) Callan-Symanzik equation above. Let us attempt to compute the integral in the exponent which multiplies $\Gamma^{(4)}(\overline{g})$. Using $\overline{g}$ from our work above, we see that

$$\int_M^p d\log(p'/M)\gamma(\overline{g}(p';g)) = \int_M^p d\log(p'/M)\frac{\gamma_1}{16\pi^2}\frac{g^3}{(1 + g^2\frac{\beta_1}{8\pi^2}\log(p'/M))^{3/2}},$$
$$= \frac{8\pi^2}{\beta_1 g^2}\frac{\gamma_1}{16\pi^2}\frac{-2g^3}{(1 + g^2\frac{\beta_1}{8\pi^2}\log(p'/M))^{1/2}}\Bigg|_M^p,$$
$$= -\frac{\gamma_1 g}{\beta_1}\left[\frac{1}{(1 + g^2\frac{\beta_1}{8\pi^2}\log(p/M))^{1/2}} - 1\right],$$
$$\underset{p\to\infty}{\approx} \frac{\gamma_1 g}{\beta_1}.$$

Unfortunately, this result cannot be trusted in general. This is because a very large portion of this integral came from the lower bound $p' = M$ as $p \to \infty$. The energy scale M is usually chosen to represent the beginning of the non-perturbative regime in an asymptotically free field theory so our one-loop estimate of the functions $\beta(g), \gamma(g)$ cannot be trusted near $p = M$.

However, the calculation has taught us an important lesson. Although the precise value of the integral is largely uncalculable, the form of the solution is predicted. In particular, our evaluation of the integral showed us that whatever the result will be, it will be a constant, independent of p at large momenta. Therefore, using our work from above, the general four-point function will be of the form $\Gamma^{(4)}e^{\text{constant}} \propto \overline{g}^2$. Because we know the behavior of $\overline{g}^2$ as $p/M \to \infty$, we conclude that

$$\boxed{\therefore \Gamma^{(4)} \underset{p\to\infty}{\sim} \frac{8\pi^2}{\beta_1\log(p/M)}.} \tag{2.c.1}$$

Physics 523, Quantum Field Theory II
Midterm Examination

Due Monday, 29th March 2004

Jacob Lewis Bourjaily
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1. Loop Integrals in Dimensional Regularization

We are to verify the identity

$$\int \frac{d^dq}{(2\pi)^d} \frac{(d-2n)q^2 - dm^2}{(q^2-m^2)^{n+1}} = 0.$$

Noting the results of homework 6 and the elementary properties of the Γ-function, we may proceed directly.

$$\begin{aligned}
\int \frac{d^dq}{(2\pi)^d} \frac{(d-2n)q^2 - dm^2}{(q^2-m^2)^{n+1}} &= (d-2n)\frac{(-1)^n i}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma\left(n-\frac{d}{2}\right)}{\Gamma(n+1)} \frac{1}{(m^2)^{n-d/2}} - dm^2 \frac{(-1)^{n+1} i}{(4\pi)^{d/2}} \frac{\left(n+1-\frac{d}{2}\right)}{\Gamma(n+1)} \frac{1}{(m^2)^{n+1-d/2}}, \\
&= \frac{(-1)^n i}{(4\pi)^{d/2}} \frac{d}{\Gamma(n+1)} \left[(d/2-n) \frac{\Gamma\left(n-\frac{d}{2}\right)}{(m^2)^{n-d/2}} + m^2 \frac{\Gamma\left(n+1-\frac{d}{2}\right)}{(m^2)^{n+1-d/2}} \right], \\
&= \frac{(-1)^n i}{(4\pi)^{d/2}} \frac{d}{\Gamma(n+1)} \frac{1}{(m^2)^{n-d/2}} \left[-(n-d/2)\Gamma(n-d/2) + \Gamma(n+1-d/2) \right], \\
&= \frac{(-1)^n i}{(4\pi)^{d/2}} \frac{d}{\Gamma(n+1)} \frac{1}{(m^2)^{n-d/2}} \left[-\Gamma(n+1-d/2) + \Gamma(n+1-d/2) \right], \\
&= 0.
\end{aligned}$$

$$\boxed{\therefore \int \frac{d^dq}{(2\pi)^d} \frac{(d-2n)q^2 - dm^2}{(q^2-m^2)^{n+1}} = 0.} \tag{1.a}$$

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Let us now evaluate the following loop integral,

$$I(p^2, m_1^2, m_2^2) = -ie^2 \int \frac{d^dq}{(2\pi)^d} \frac{1}{((q+p/2)^2 - m_1^2 + i\epsilon)((q-p/2)^2 - m_2^2 + i\epsilon)}.$$

To evaluate this integral lucidly, let us first introduce the change of variables $k \equiv q + p/2$. Introducing the Feynman parameter x, the integral becomes,

$$\int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{1}{[x((k-p)^2 - m_2^2 + i\epsilon) + (1-x)(k^2 - m_1^2 + i\epsilon)]^2}.$$

Introducing the variables,

$$\ell \equiv k - xp \qquad \text{and} \qquad \Delta \equiv x(x-1)p^2 + xm_2^2 + (1-x)m_1^2,$$

we see that

$$\begin{aligned}
I(p^2, m_1^2, m_2^2) &= \int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{[\ell^2 - \Delta + i\epsilon]^2}, \\
&= \int_0^1 dx \left[\frac{i}{(4\pi)^{d/2}} \frac{\Gamma\left(2-\frac{d}{2}\right)}{\Delta^{2-d/2}} \right], \\
&\underset{d\to 4}{\sim} \frac{i}{(4\pi)^2} \int_0^1 dx \left[\frac{2}{\epsilon} - \log\Delta - \gamma_E + \log(4\pi) + \mathcal{O}(\epsilon) \right]
\end{aligned}$$

$$\boxed{\therefore I(p^2, m_1^2, m_2^2) \underset{d\to 4}{\sim} \frac{i}{(4\pi)^2} \int_0^1 dx \left[\frac{2}{\epsilon} + \log \frac{1}{x(x-1)p^2 + xm_2^2 + (1-x)m_1^2} - \gamma_E + \log(4\pi) \right].} \tag{1.b}$$

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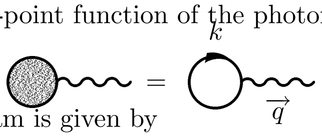
The One-Loop Structure of Quantum Electrodynamics

While studying the superficial divergences of quantum electrodynamics, we noted that gauge invariance–and hence the Ward identity–made several superficially divergent diagrams either converge or vanish. We are to verify these claims explicitly.

Superficially, the one-point function of the photon has a cubic divergence. Let us demonstrate that in fact, to the one-loop order, the one-point function of the photon vanishes.

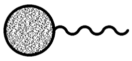
To one-loop order, we see that



The amplitude for the above diagram is given by

$$\begin{aligned} i\mathscr{M} &= (-1)\epsilon^*_\mu(q)\int\frac{d^dk}{(2\pi)^d}\text{Tr}\left[\frac{i(\not{k}+m_e)}{(k^2-m_e^2+i\epsilon)}(-ie\gamma^\mu)\right], \\ &= -\epsilon^*_\mu(q)e\int\frac{d^dk}{(4\pi)^d}\frac{\text{Tr}\left(\not{k}\gamma^\mu+m\gamma^\mu\right)}{(k^2-m_e^2+i\epsilon)}, \\ &= -\epsilon^*_\mu(q)4e\int\frac{d^dk}{(4\pi)^d}\frac{k^\mu}{(k^2-m_e^2+i\epsilon)}, \\ &= 0. \end{aligned}$$

Therefore, to one-loop order,



$= 0.$

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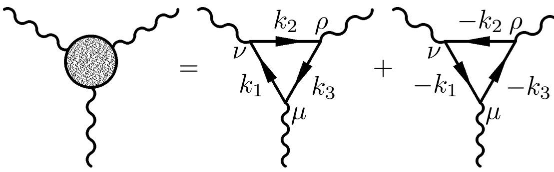
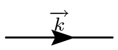
Similarly, we argued that although the photon three-point function has a superficial, linear divergence, its amplitude should also vanish. Let us now demonstrate this fact.

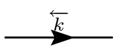
To one-loop order, we see that



Note that the second diagram has been labeled the same as the first diagram but with relative minus signs on the momenta k. This is because the Feynman propagator has the property that



$$= \frac{i(\not{k}+m_e)}{(k^2-m_e^2+i\epsilon)} \quad \text{whereas}$$



$$= \frac{i(-\not{k}+m_e)}{(k^2-m_e^2+i\epsilon)}.$$

Let us consider the evaluation of the first diagram. Its amplitude is proportional to integration over

$$\text{Tr}\left[\gamma^\mu(\not{k}_1+m_e)\gamma^\nu(\not{k}_2+m_e)\gamma^\rho(\not{k}_3+m_e)\right].$$

Because only those traces over an even number of γ-matrices are non-vanishing, this is equal to

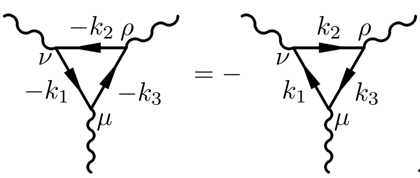
$$\text{Tr}\left[\gamma^\mu\not{k}_1\gamma^\nu\not{k}_2\gamma^\rho\not{k}_3\right]+m_e^2\left(\text{Tr}\left[\gamma^\mu\not{k}_1\gamma^\nu\gamma^\rho\right]+\text{Tr}\left[\gamma^\mu\gamma^\nu\not{k}_2\gamma^\rho\right]\text{Tr}\left[\gamma^\mu\gamma^\nu\gamma^\rho\not{k}_3\right]\right).$$

Notice that the only remaining traces involve an odd number of momenta k.

Similarly, we see that the amplitude of the second diagram is proportional to integration over

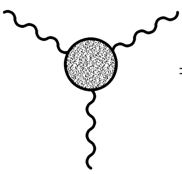
$$\text{Tr}\left[(-\not{k}_3+m_e)\gamma^\rho(-\not{k}_2+m_e)\gamma^\nu(-\not{k}_1+m_e)\gamma^\mu\right]=-\text{Tr}\left[\not{k}_3\gamma^\rho\not{k}_2\gamma^\nu\not{k}_1\gamma^\mu\right]-m_e^2\left(\text{Tr}\left[\not{k}_3\gamma^\rho\gamma^\nu\gamma^\mu\right]+\text{Tr}\left[\gamma^\rho\not{k}_2\gamma^\nu\gamma^\mu\right]\text{Tr}\left[\gamma^\rho\gamma^\nu\not{k}_1\gamma^\mu\right]\right).$$

But, noting identity (5.7) of Peskin and Schroeder, the traces of each expression are equal. Therefore, the negative contribution from the second diagram cancels the contribution from the first.



.

Therefore, to one-loop order,



$= 0.$

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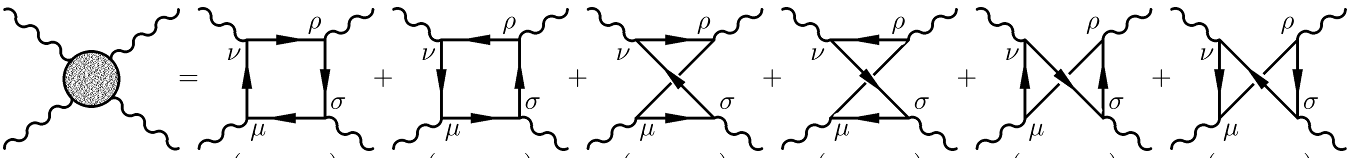
Lastly, our analysis showed that the photon four-point function has a logarithmic, superficial divergence, but by gauge invariance this amplitude is convergent. We are to demonstrate that the photon four-point function does not diverge to the one-loop order in perturbation theory.

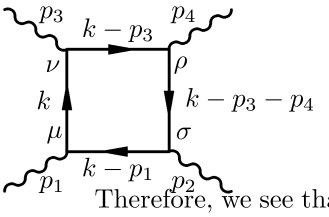
To one-loop order, we see that

I **II** **III** **IV** **V** **VI**



$(\mu\ \nu\ \rho\ \sigma)$ $(\sigma\ \rho\ \nu\ \mu)$ $(\mu\ \sigma\ \nu\ \rho)$ $(\rho\ \nu\ \sigma\ \mu)$ $(\mu\ \nu\ \sigma\ \rho)$ $(\rho\ \sigma\ \nu\ \mu)$

Because it is our task to demonstrate that the above amplitude converges–rather than actually compute the amplitude–we may make several helpful simplifications. To illustrate the first major simplification, let us analyze the first diagram, (I).



$$= -e^4\int\frac{d^dk}{(2\pi)^d}\frac{\mathrm{Tr}\left[(\not k-\not p_1+m_e)\gamma^\mu(\not k+m_e)\gamma^\nu(\not k-\not p_3+m_e)\gamma^\rho(\not k-\not p_3-\not p_4+m_e)\gamma^\sigma\right]}{((k-p_1)^2-m_e^2)(k^2-m_e^2)((k-p_3)^2-m_e^2)((k-p_3-p_4)^2-m_e^2)},$$
$$= -e^4\int\frac{d^dk}{(2\pi)^d}\frac{\mathrm{Tr}\left[\not k\gamma^\mu\not k\gamma^\nu\not k\gamma^\rho\not k\gamma^\sigma\right]}{(k^2-m_e^2)^4}\ +\ \text{finite terms.}$$

Therefore, we see that the divergent part of each diagram is a function of *only the order* of γ-matrices in the trace.

Now, we claim that the divergence of diagram (I) is the same as (II), (III)∼(IV), and (V)∼(VI). First, note that the relative change of sign for the loop momentum k between each pair will not change the divergence of the diagram because each involves only $k^4 = (-k)^4$. Secondly, the ordering of the vertices are precisely reversed for each pair and so by identity (5.7) of Peskin and Schroeder they are equal. Therefore the total divergence of these six diagrams will be twice that of (I), (III), and (V) alone.

Let us continue to compute the divergence of diagram (I) before illustrating the sum of all six diagrams. Because, as we will show, the sum of the diagrams will converge, we will continue without dimensional regularization.[1]

In our calculation below, we will repeatedly make use of γ-matrix algebra proved in homework (including that of semester I). Also, note our use of identity (A.42) from Peskin and Schroeder. Let us begin to evaluate the divergence of diagram (I). The integrand is proportional to

$$\begin{aligned}
\mathrm{Tr}\left[\not k\gamma^\mu\not k\gamma^\nu\not k\gamma^\rho\not k\gamma^\sigma\right] &= k_\alpha k_\beta k_\gamma k_\delta \mathrm{Tr}\left[\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu\gamma^\gamma\gamma^\rho\gamma^\delta\gamma^\sigma\right],\\
&\to \frac{1}{d(d+2)}(k^2)^2\left(g_{\alpha\beta}g_{\gamma\delta}+g_{\alpha\gamma}g_{\beta\delta}+g_{\alpha\delta}g_{\beta\gamma}\right)\mathrm{Tr}\left[\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu\gamma^\gamma\gamma^\rho\gamma^\delta\gamma^\sigma\right],\\
&\propto \mathrm{Tr}[\gamma\gamma^\mu\gamma\,\gamma^\nu\,\gamma\gamma^\rho\gamma\,\gamma^\sigma]+\mathrm{Tr}[\gamma\gamma^\mu\gamma\gamma^\nu\gamma\gamma^\rho\gamma\,\gamma^\sigma]+\mathrm{Tr}[\gamma\gamma^\mu\gamma\gamma^\nu\gamma\gamma^\rho\gamma\,\gamma^\sigma],\\
&= \mathrm{Tr}[(-2\gamma^\mu)\gamma^\nu(-2\gamma^\rho)\gamma^\sigma]+\mathrm{Tr}[(-2)\gamma^\nu\gamma\gamma^\mu\gamma^\rho\gamma\,\gamma^\sigma]+\mathrm{Tr}[\gamma\gamma^\mu(-2\gamma^\nu)\gamma^\rho\gamma\,\gamma^\sigma],\\
&= 4\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma]-2\mathrm{Tr}[\gamma^\nu 4g^{\mu\rho}\gamma^\sigma]-2\mathrm{Tr}[-2\gamma^\rho\gamma^\nu\gamma^\mu\gamma^\sigma],\\
&= 8\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma]-8g^{\mu\rho}\mathrm{Tr}[\gamma^\nu\gamma^\sigma],\\
&= 32\left(g^{\rho\sigma}g^{\mu\nu}-g^{\nu\sigma}g^{\mu\rho}+g^{\mu\sigma}g^{\nu\rho}\right)-32g^{\mu\rho}g^{\nu\sigma},\\
&\propto \left(g^{\mu\nu}g^{\rho\sigma}-2g^{\mu\rho}g^{\nu\sigma}+g^{\mu\sigma}g^{\nu\rho}\right).
\end{aligned}$$

Therefore, when we evaluate the amplitude for all six diagrams, the divergent integral will be over a term proportional to $(g^{\mu\nu}g^{\rho\sigma}-2g^{\mu\rho}g^{\nu\sigma}+g^{\mu\sigma}g^{\nu\rho})$ together with the analogous terms under the other two distinct permutations. Therefore, the amplitude's divergence will be proportional to,

$$(g^{\mu\nu}g^{\rho\sigma}-2g^{\mu\rho}g^{\nu\sigma}+g^{\mu\sigma}g^{\nu\rho})+(g^{\mu\rho}g^{\nu\sigma}-2g^{\mu\nu}g^{\rho\sigma}+g^{\mu\sigma}g^{\rho\nu})+(g^{\mu\nu}g^{\rho\sigma}-2g^{\mu\sigma}g^{\nu\rho}+g^{\mu\rho}g^{\nu\sigma})=0.$$

Therefore, the photon's four-point function is convergent to loop-order in QED. ὅπερ ἔδει δεῖξαι

[1]It is easier for our purposes to work with $d = 4$ trace-algebra. Because the total divergence will vanish in $d = 4$, it must also vanish in general dimensional regularization.

The β-function of Quantum Chromodynamics

We are given that, at one-loop order in perturbation theory, the divergent parts of the counter terms of quantum chromodynamics are

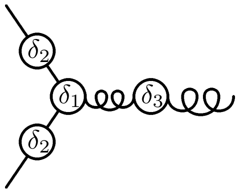
$$\delta_1 = -\frac{7}{2}\frac{g^2}{(4\pi)^2}\log\frac{\Lambda^2}{M^2}, \quad \delta_2 = -\frac{1}{2}\frac{g^2}{(4\pi)^2}\log\frac{\Lambda^2}{M^2}, \quad \text{and} \quad \delta_3 = \left(5 - \frac{2}{3}n_f\right)\frac{g^2}{(4\pi)^2}\log\frac{\Lambda^2}{M^2},$$

where the δ_i are defined in analogy to quantum electrodynamics. We see that these directly imply that

$$B_g = \frac{7}{2}\frac{g^2}{(4\pi)^2}, \quad A_f = -\frac{1}{2}\frac{g^2}{(4\pi)^2}, \quad \text{and} \quad A_{gl} = \left(5 - \frac{2}{3}n_f\right)\frac{g^2}{(4\pi)^2},$$

where A_f corresponds to fermion self-energy and A_{gl} corresponds to gluon self-energy.

Let us now compute the β-function for the strong coupling g. This corresponds to the diagram,



Therefore, because $\beta_g = -2gB_g - 2gA_F - gA_{gl}$, we see that

$$\boxed{\therefore \beta_g = -\left(11 - \frac{2}{3}n_f\right)\frac{g^3}{16\pi^2}.} \tag{3.a}$$

In homework 10, we computed the general running coupling constat associated with quantum chromodynamics. To relate that result with our work here, we should set the undetermined constant β_1 to $\left(11 - \frac{2}{3}n_f\right)$. So from our results of homework 10, we see that the square of the running coupling $\overline{g}$ is

$$\boxed{\therefore \overline{g}^2 = \frac{g^2}{1 + \frac{g^2}{8\pi^2}\left(11 - \frac{2}{3}n_f\right)\log(p/M)}.} \tag{3.b}$$

We see that the coupling constant will be asymptotically free if $11 > 2/3n_f$. This is because asymptotic freedom is directly a result of a negative β-function. It is clear that $\beta_g < 0$ only if $n_f < 33/2 = 16.5$. Also, again by the results of homework 10, we see that at large energy ($p/M \to \infty$), the square of the coupling constant can be approximated by

$$\boxed{\overline{g}^2 \underset{\frac{p}{M}\to\infty}{\approx} \frac{8\pi^2}{\left(11 - \frac{2}{3}n_f\right)\log(p/M)}.} \tag{3.c}$$